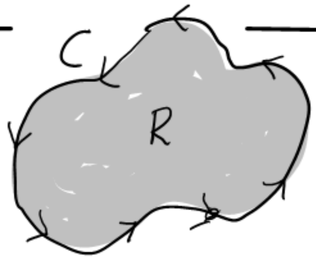
Section 16.4 Green's Theorem (Continued)

Recall:

Green's Theorem



Suppose $\vec{r}(t)$ is a piecewise smooth curve C on the plane enclosing a region R, and $F = \langle M, N \rangle$ is a vector with continuous first partial derivatives in an open region containing R. Then:

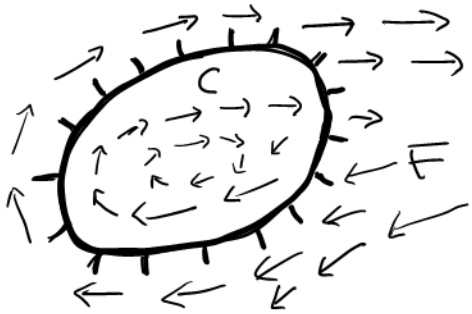
$$\begin{pmatrix}\text{outward flux}\\ \text{across } C\end{pmatrix} = \oint_C F \cdot \vec{n}\, ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

Today's Goal: Investigate an alternate form of Greens Theorem:

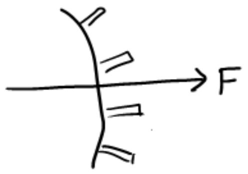
$$\begin{pmatrix}\text{Counterclockwise}\\ \text{circulation}\\ \text{around } C\end{pmatrix} = \oint F \cdot T\, ds = \oint M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Questions: What is "circulation"? Why is this theorem true?

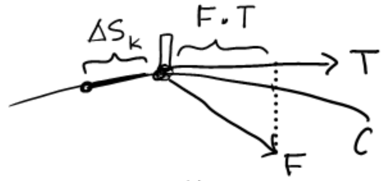
Imagine C as a "chain" with little paddle wheels, and F represents the velocity of a fluid flowing on the plane. Then F may cause the chain of paddle wheels to spin.



Strong contribution to circulation



weak contribututation



Force on k^{th} paddle wheel proportional to $F \cdot T\, \Delta S_k$

Circulation or "curl" around curve C is

$$\lim_{|P| \to 0} \sum_{k=1}^{n} F \cdot T\, \Delta S_k = \oint_C F \cdot T\, ds = \oint_C F \cdot \frac{v(t)}{|v(t)|}\, |v(t)|\, dt$$

$$= \oint_C F \cdot \frac{d\vec{r}}{dt}\, dt = \oint_C \langle M, N \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt = \oint M\,dx + N\,dy$$

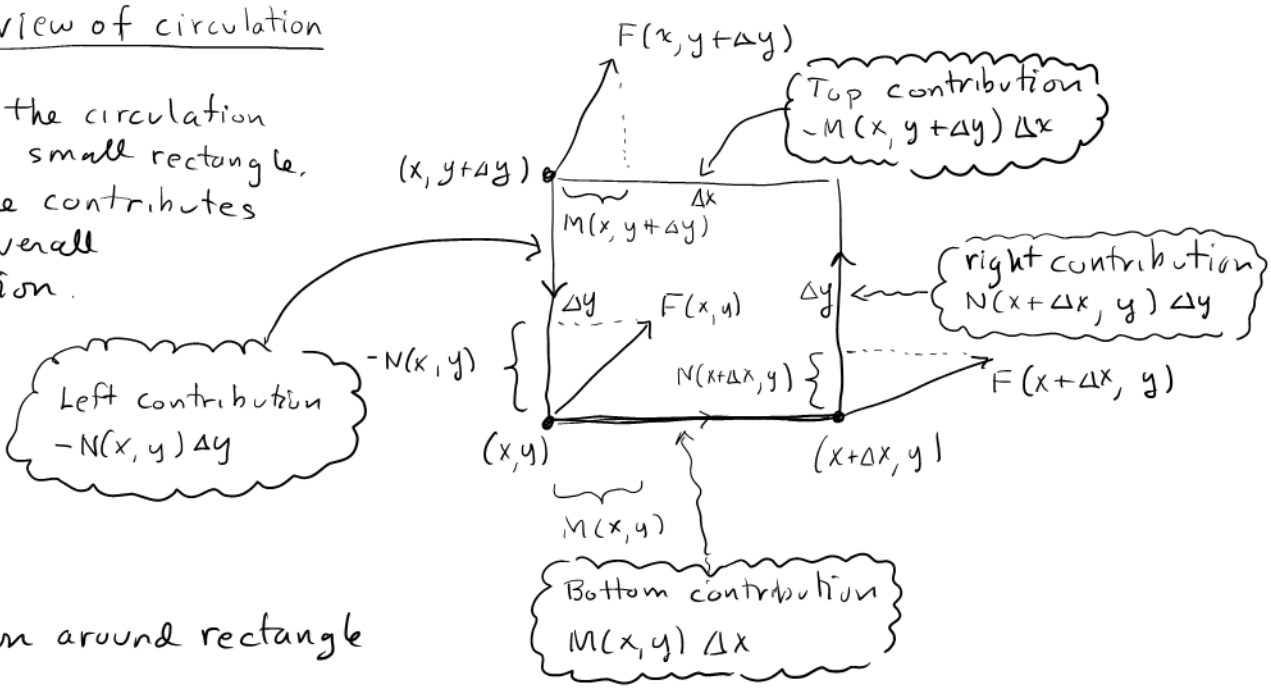
Thus:

$$\boxed{\begin{pmatrix}\text{circulation}\\ \text{around } C\end{pmatrix} = \oint_C F \cdot T\, ds = \oint_C M\,dx + N\,dy}$$

First half of alternate form of Green's Theorem

Another view of circulation

Consider the circulation around a small rectangle. Each side contributes to the overall circulation.

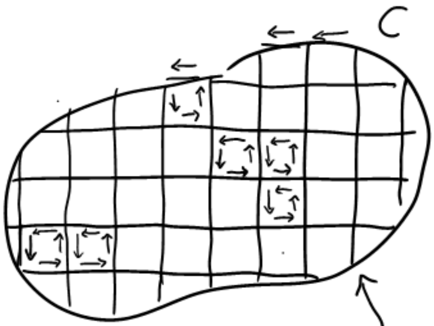


Circulation around rectangle

$$\approx N(x+\Delta x, y)\Delta y - N(x,y)\Delta y \quad - M(x, y+\Delta y)\Delta x + M(x,y)\Delta x$$

$$= \frac{N(x+\Delta x, y) - N(x,y)}{\Delta x}\Delta x\,\Delta y - \frac{M(x, y+\Delta y) - M(x,y)}{\Delta y}\Delta x\,\Delta y$$

$$\approx \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{\text{"curl" - measures circulation at } (x,y)\text{. More on this soon}} \Delta x\,\Delta y$$



Now chop R into small rectangles

$$\left(\text{Circulation around } C\right) \approx \sum_{k=1}^{n}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x_k\,\Delta y_k$$

Note: contributions along edges shared by adjacent rectangles cancel. Only uncancelled contribution is along boundary

$$\begin{pmatrix}\text{Circulation}\\ \text{around } C\end{pmatrix} = \lim_{|P|\to 0}\sum_{k=1}^{n}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x_k\,\Delta y_k$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

We've now computed circulation around C in two ways, and this gives the alternate form of Greens Theorem:

$$\boxed{\begin{pmatrix}\text{Circulation}\\ \text{around } C\end{pmatrix} = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

Divergence and Curl

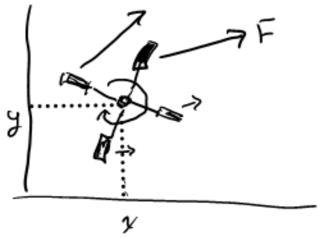
The following ideas are significant concepts in Green's Theorem

Let $F(x,y) = \langle M(x,y), N(x,y)\rangle = \langle M, N\rangle$ represent velocity of a fluid or gas

$$\operatorname{div} F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \left(\begin{array}{l}\text{measure of compression} \\ \text{or expansion at } (x,y)\end{array}\right)$$

$$\operatorname{curl} F = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \left(\begin{array}{l}\text{measure of circulation} \\ \text{at any point } (x,y)\end{array}\right)$$

For curl F, think of inserting a "paddle wheel" at (x,y). Then curl F measures the wheel's spin.

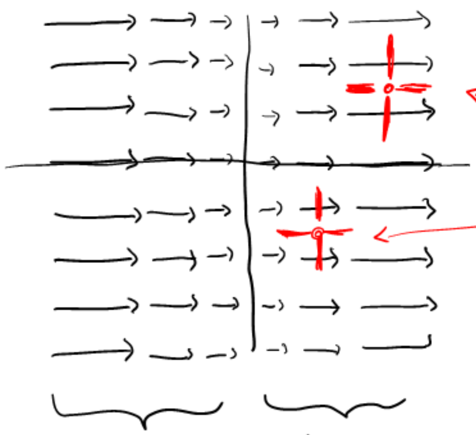


curl F > 0 ⟷ counterclockwise spin
curl F < 0 ⟷ clockwise spin.
curl F = 0 ⟷ no spin.

Example $F = \langle x^2, 0\rangle$

$$\operatorname{div} F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 2x + 0 = 2x$$

$$\operatorname{curl} F = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0 + 0 = 0$$



At each point (x,y) curl $F = 0$
paddle wheels locked – no spin

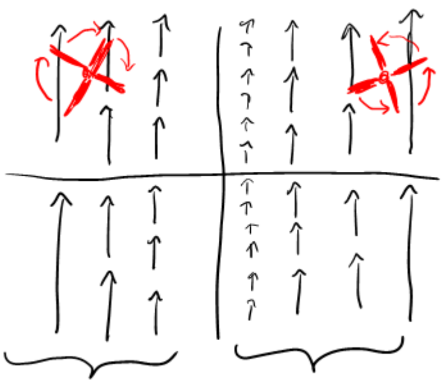
$\operatorname{div} F = 2x < 0$ (compression) $\qquad \operatorname{div} F = 2x > 0$ (expansion)

Example $F = \langle 0, x^2\rangle$

$$\operatorname{div} F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0 + 0 = 0$$

$\therefore$ No compression

$$\operatorname{curl} F = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x + 0 = 2x$$



$\operatorname{curl} F = 2x < 0$ clockwise $\qquad \operatorname{curl} F = 2x > 0$ counterclockwise

See other examples in text

Intuitive View of Green's Theorem

Greens Theo. First Form	$\left(\begin{array}{l}\text{flux accross } C \\ \text{i.e. divergence} \\ \text{in region } R\end{array}\right) = \iint_R \operatorname{div} F \, dA$
Greens Theo Second form	$\left(\begin{array}{l}\text{circulation or} \\ \text{curl around } C\end{array}\right) = \iint_R \operatorname{curl} F \, dA$