6. (14 pts.) Find the radian measure of the angle formed by the vectors $\langle 2,1,1\rangle$ and $\langle \sqrt{3}, 0, \sqrt{3}\,\rangle$.

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}\right)$$

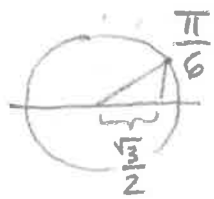
$$= \cos^{-1}\left(\frac{\langle 2,1,1\rangle \cdot \langle \sqrt{3}, 0, \sqrt{3}\rangle}{\sqrt{2^2+1^2+1^2}\,\sqrt{\sqrt{3}^2+0^2+\sqrt{3}^2}}\right)$$

$$= \cos^{-1}\left(\frac{3\sqrt{3}}{\sqrt{6}\,\sqrt{6}}\right)$$

$$= \cos^{-1}\left(\frac{3\sqrt{3}}{6}\right)$$

$$= \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \boxed{\frac{\pi}{6}}$$



Good Luck!

VCU

MATH 307

Multivariate Calculus

R. Hammack

Test 1

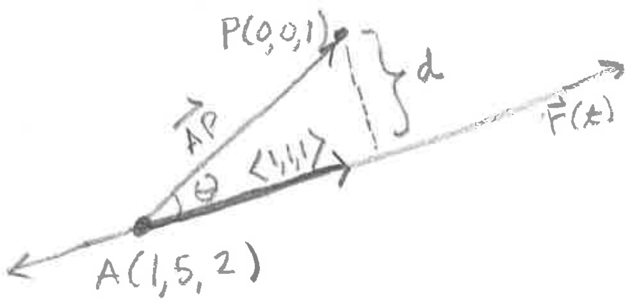
February 5, 2014

Name: Richard

Score: 100

Directions. Solve the following questions in the space provided. Unless noted otherwise, you must show your work to receive full credit. This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Put a your final answer in a $\boxed{\text{box,}}$ where appropriate.

1. (14 pts.) Find the distance between the point $P(0,0,1)$ and the line $\mathbf{r}(t) = \langle 1,5,2\rangle + t\langle 1,1,1\rangle$.



$\vec{AP} = \langle -1, -5, -1\rangle$

$\vec{AP} \times \langle 1,1,1\rangle = \begin{vmatrix} i & j & k \\ -1 & -5 & -1 \\ 1 & 1 & 1 \end{vmatrix} =$

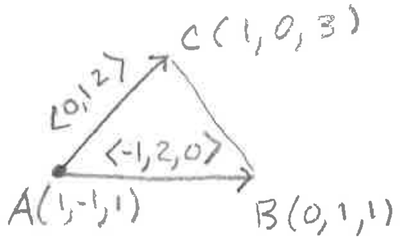
$\langle -4, 0\ 4\rangle$

From the above picture, $d = |\vec{AP}|\sin\theta$

$$= \frac{|\vec{AP}|\,|\langle 1,1,1\rangle|\cos\theta}{|\langle 1,1,1\rangle|} = \frac{|\vec{AP} \times \langle 1,1,1\rangle|}{\sqrt{1^2+1^2+1^2}}$$

$$= \frac{|\langle -4,0,4\rangle|}{\sqrt{3}} = \frac{\sqrt{4^2+4^2}}{\sqrt{3}} = \frac{\sqrt{32}}{\sqrt{3}} = \boxed{\frac{4\sqrt{2}}{\sqrt{3}} \text{ units}}$$

2. (14 pts.) Find the area of the triangle with vertices $A(1,-1,1)$, $B(0,1,1)$, $C(1,0,3)$.



$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{vmatrix} = \langle -4, -2, 1 \rangle$$

$$\text{Area} = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right| = \frac{1}{2}\left|\langle -4, 2, 1\rangle\right|$$

$$= \frac{1}{2}\sqrt{4^2 + 2^2 + 1^2}$$

$$= \frac{1}{2}\sqrt{21} = \boxed{\frac{\sqrt{21}}{2} \text{ square units}}$$

3. This page concerns the line $x = 2t+1$, $y = 3t+2$, $z = 4t+3$, as well as the line $x = s+2$, $y = 2s+4$, $z = -4s-1$.

(a) (10 pts.) These lines intersect at a point. Find it. Say they cross at $\langle x, y, z\rangle$. Then:

$$\begin{aligned} x &= 2t+1 = s+2 \\ y &= 3t+2 = 2s+4 \\ z &= 4t+3 = -4s-1 \end{aligned} \quad \xrightarrow{\text{Get System}} \quad \begin{aligned} 2t - s &= 1 \\ 3t - 2s &= 2 \\ 4t + 4s &= -4 \end{aligned} \quad \longrightarrow \quad \begin{aligned} 2t - s &= 1 \\ 3t - 2s &= 2 \\ t + s &= -1 \end{aligned}$$

Add 3rd equation to 1st an get $2t = 0 \Rightarrow t = 0$

Therefore point of intersection is $\langle 2\cdot 0+1, 3\cdot 0+2, 4\cdot 0+3\rangle = \boxed{\langle 1,2,3\rangle}$

(b) (10 pts.) Note that the vector $\langle -20, 12, 1\rangle$ is orthogonal to both lines.
Use this information to find an equation of the plane containing the two lines.

$$-20x + 12y + z = -20(1) + 12(2) + 3$$

$$\boxed{-20x + 12y + z = 7}$$

(c) (10 pts.) Find the point where the line $r(t) = \langle t, 2t, 3t\rangle$ intersects the plane from part (b), above.

At intersection we have $-20t + 12(2t) + 3t = 7$ so $7t = 7$ or $t = 1$. Thus intersection point is $\vec{r}(1) = \langle 1, 2, 3\rangle$

4. (14 pts.) Find the length of the following curve:
$\mathbf{r}(t) = \langle e^t \cos t,\ e^t,\ e^t \sin t\rangle$, where $-\ln(4) \leqslant t \leqslant 0$.

$$L = \int_{-\ln 4}^{0} \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt$$

$$= \int_{-\ln 4}^{0} \sqrt{(e^t\cos t - e^t \sin t)^2 + (e^t)^2 + (e^t \sin t + e^t \cos t)^2}\, dt$$

$$= \int_{-\ln 4}^{0} \sqrt{e^{2t}\cos^2 t - 2e^{2t}\cos t \sin t + e^{2t}\sin^2 t + e^{2t} + e^{2t}\sin^2 t + 2e^{2t}\sin t \cos t + e^{2t}\cos^2 t}\, dt$$

$$= \int_{-\ln 4}^{0} \sqrt{2e^{2t}\cos^2 t + 2e^{2t}\sin^2 t + e^{2t}}\, dt$$

$$= \int_{-\ln 4}^{0} e^t \sqrt{2\cos^2 t + 2\sin^2 t + 1}\, dt = \int_{-\ln 4}^{0} e^t \sqrt{2(\cos^2 t + \sin^2 t) + 1}\, dt$$

$$= \int_{-\ln 4}^{0} e^t \sqrt{3}\, dt = \left[\sqrt{3}\, e^t\right]_{-\ln 4}^{0} = \sqrt{3}\, e^0 - \sqrt{3}\, e^{-\ln 4}$$

$$= \sqrt{3} - \frac{\sqrt{3}}{e^{\ln 4}} = \sqrt{3} - \frac{\sqrt{3}}{4} = \boxed{\frac{3\sqrt{3}}{4} \text{ units}}$$

5. (14 pts.) Find $\mathbf{r}(t)$ if $\frac{d\mathbf{r}}{dt} = \left\langle \frac{2}{3}\sqrt{t+1},\ e^{-t},\ \frac{1}{t+1}\right\rangle = \left\langle \frac{2}{3}(t+1)^{\frac{1}{2}},\ e^{-t},\ \frac{1}{t+1} \right\rangle$
and $\mathbf{r}(0) = \langle 0, 0, 1\rangle$.

$$\vec{r}(t) = \int \frac{dr}{dt}\, dt = \left\langle \int \frac{2}{3}(t+1)^{\frac{1}{2}} dt,\ \int e^{-t} dt,\ \int \frac{1}{t+1}\, dt \right\rangle$$

$$= \left\langle \frac{4}{9}(t+1)^{3/2} + C_1,\ -e^{-t} + C_2,\ \ln|t+1| + C_3 \right\rangle$$

Now,

$$\langle 0,0,1\rangle = \vec{r}(0) = \left\langle \frac{4}{9}\sqrt{0+1}^{\,3} + C_1,\ -e^0 + C_2,\ \ln|0+1| + C_3 \right\rangle$$

$$= \left\langle \frac{4}{9} + C_1,\ -1 + C_2,\ 0 + C_3 \right\rangle$$

Therefore $C_1 = -\frac{4}{9}$, $C_2 = 1$, $C_3 = 1$

$$\boxed{\text{Thus } \vec{r}(t) = \left\langle \frac{4}{9}\sqrt{t+1}^{\,3} - \frac{4}{9},\ -e^{-t} + 1,\ \ln|t+1| + 1 \right\rangle}$$