VCU

MATH 307

MULTIVARIATE CALCULUS

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SAMPLE TEST 1

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Name: Richard

Score: 100

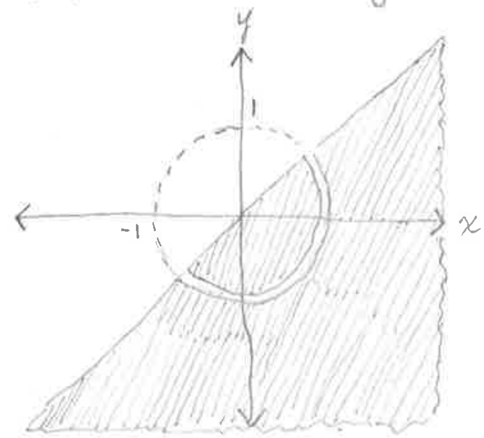
Directions. Solve the following questions in the space provided. Unless noted otherwise, you must show your work to receive full credit. This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Put a your final answer in a $\boxed{\text{box,}}$ where appropriate.

6. (10 pts.) Suppose $f(x, y) = \dfrac{\sqrt{x-y}}{1-x^2-y^2}$.
Sketch the domain of this function.

Must have $x - y \geq 0 \rightsquigarrow y \leq x$

and $1 - x^2 - y^2 \neq 1 \rightsquigarrow x^2 + y^2 \neq 1$

Thus any point (x, y) in the domain is below the line $y = x$ and not on the unit circle. This region is sketched:



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1. (24 points) Let $\mathbf{u} = \langle 2, -2, 3 \rangle$ and $\mathbf{v} = \langle 0, -2, 1 \rangle$.

(a) $\mathbf{u} \cdot \mathbf{v} = 2 \cdot 0 + (-2)(-2) + 3 \cdot 1 = \boxed{7}$

(b) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ 2 & -2 & 3 \\ 0 & -2 & 1 \end{vmatrix} = \boxed{\langle 4, -2, -4 \rangle}$

(c) $|\mathbf{u}| = \sqrt{2^2 + (-2)^2 + 3^2} = \boxed{\sqrt{17}}$

(d) $|\mathbf{v}| = \sqrt{0^2 + (-2)^2 + 1^2} = \boxed{\sqrt{5}}$

(e) Find $\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Because $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$, we have

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{7}{\sqrt{17}\sqrt{5}} = \boxed{\frac{7}{\sqrt{85}}}$$

(f) Find $\mathbf{x}$, where $2\mathbf{x} - \mathbf{v} = 3\mathbf{u}$.

$$\vec{x} = \frac{1}{2}(3\vec{u} + \vec{v}) = \frac{3}{2}\vec{u} + \frac{1}{2}\vec{v} = \frac{3}{2}\langle 2, -2, 3 \rangle + \frac{1}{2}\langle 0, -2, 1 \rangle$$

$$= \langle 3, -3, \tfrac{9}{2} \rangle + \langle 0, -1, \tfrac{1}{2} \rangle = \boxed{\langle 3, -4, 5 \rangle}$$

2. (10 pts.) Find the equation for the plane containing the point $(1, 4, 2)$ and the line $\mathbf{r}(t) = (1 - 2t)\mathbf{i} + (2 + t)\mathbf{j} + (5 - t)\mathbf{k}$.

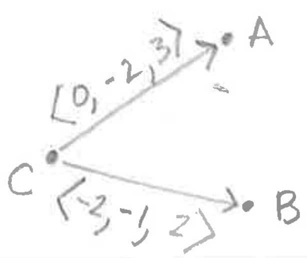
$= \langle 1-2t,\ 2+t,\ 5-t \rangle$

Here are two points on the line:

For $t=0$: $A(1, 2, 5)$

For $t=1$: $B(-1, 3, 4)$

These two points are on the plane, and so is $C(1, 4, 2)$



Thus a normal vector to the plane is $\vec{n} = \overrightarrow{CA} \times \overrightarrow{CB} =$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -2 & 3 \\ -2 & -1 & 2 \end{vmatrix} = \langle -1, -6, -4 \rangle.$$

We can scale this by -1 to get normal vector $\langle 1, 6, 4 \rangle$

Equation for plane is then

$1x + 6y + 4z = 1\cdot 1 + 6\cdot 4 + 4\cdot 2$

$$\boxed{x + 6y + 4z = 33}$$

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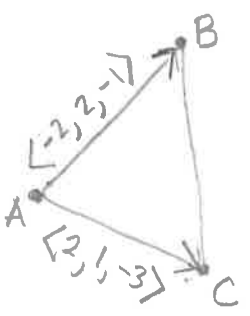
3. (16 pts.) Consider the triangle in space whose vertices are the points $A(1, 1, 4)$, $B(-1, 3, 3)$ and $C(3, 2, 1)$.

(a) Find a vector normal to the plane that the triangle lies in.



$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & -1 \\ 2 & 1 & -3 \end{vmatrix} = \langle -5, -8, -6 \rangle$$

That's an OK answer, but we can get rid of the negatives by scaling by -1. Normal vector: $\boxed{\langle 5, 8, 6 \rangle}$

(b) Find the area of the triangle ABC.

$$\text{Area} = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} |\langle -5, -8, -6 \rangle|$$

$$= \frac{1}{2}\sqrt{(-5)^2 + (-8)^2 + (-6)^2} = \frac{1}{2}\sqrt{25 + 64 + 36}$$

$$= \frac{1}{2}\sqrt{125} = \frac{1}{2}\sqrt{25\cdot 5} = \boxed{\frac{5\sqrt{5}}{2} \text{ square units}}$$

4. (30 pts.)

(a) Find a (non-zero) vector orthogonal to $\mathbf{v} = \langle 5, 4, -7 \rangle$.

There are numerous easy answers, such as $\langle -4, 5, 0\rangle$ or $\langle 7, 0, 5\rangle$ or $\langle 0, 7, 4\rangle$ (each dotted with $\vec{v}$ is 0, so they are all orthogonal to $\vec{v}$.

(b) $\int_{\pi/4}^{\pi} \langle \sin t, 1, \sin t \cos t \rangle \, dt = \left[\langle -\cos t, t, \frac{1}{2}\sin^2 t \rangle \right]_{\pi/4}^{\pi}$

$= \langle -\cos\pi, \pi, \frac{1}{2}\sin^2\pi \rangle - \langle -\cos\frac{\pi}{4}, \frac{\pi}{4}, \frac{1}{2}\sin^2\frac{\pi}{4} \rangle$

$= \langle 1, \pi, 0 \rangle - \langle -\frac{1}{\sqrt{2}}, \frac{\pi}{4}, \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2 \rangle = \boxed{\langle 1 + \frac{1}{\sqrt{2}}, \frac{3\pi}{4}, -\frac{1}{4} \rangle}$

(c) Compute the arc length of the helix $\mathbf{r}(t) = \langle t, \sin t, \cos t \rangle$ between $t = 0$ and $t = 4\pi$.

$L = \int_0^{4\pi} \sqrt{(1)^2 + (\cos t)^2 + (-\sin t)^2} \, dt = \int_0^{4\pi} \sqrt{1+1} \, dt = \int_0^{4\pi} \sqrt{2} \, dt$

$= \left[\sqrt{2}\, t \right]_0^{4\pi} = \boxed{4\sqrt{2}\,\pi \text{ units}}$

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5. (10 pts.) An object moving in space has acceleration $\mathbf{a}(t) = \langle 1, \frac{t}{6}, 1 \rangle$ ~~feet~~ per second per second at time t seconds. Suppose that at time $t = 0$ it is at the origin and has velocity vector $\langle 1, 1, 2 \rangle$. Find the velocity function $\mathbf{v}(t)$ and its position function $\mathbf{r}(t)$.

$\vec{v}(t) = \int \vec{a}(t)\, dt = \int \langle 1, \frac{t}{6}, 1 \rangle \, dt = \langle t + c_1, \frac{t^2}{12} + c_2, t + c_3 \rangle$

But $\langle 1, 1, 2 \rangle = \vec{v}(0) = \langle c_1, c_2, c_3 \rangle$ and therefore

$$\boxed{\vec{v}(t) = \langle t + 1, \frac{t^2}{12} + 1, t + 2 \rangle}$$

Now $\vec{r}(t) = \int \vec{v}(t)\, dt = \langle \frac{t^2}{2} + t + c_1, \frac{t^3}{36} + t + c_2, \frac{t^2}{2} + 2t + c_3 \rangle$

But object is at $\langle 0, 0, 0 \rangle$ when $t = 0$, so that

$\langle 0, 0, 0 \rangle = \vec{r}(0) = \langle c_1, c_2, c_3 \rangle$.

Conclusion: $\boxed{\vec{r}(t) = \langle \frac{t^2}{2} + t, \frac{t^3}{36} + t, \frac{t^2}{2} + 2t \rangle}$