VCU

MATH 307

Multivariate Calculus

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Test 2

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Score: 100

Directions. Solve the following questions in the space provided. Unless noted otherwise, you must show your work to receive full credit. This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Put a your final answer in a $\boxed{\text{box,}}$ where appropriate.

Good Luck!

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1. (30 pts.) Consider function $z = f(x, y) = \ln(x^2 + y^2)$.

(a) State the domain of f. $\boxed{\text{All points } (x,y) \text{ in the plane except } (0,0)}$

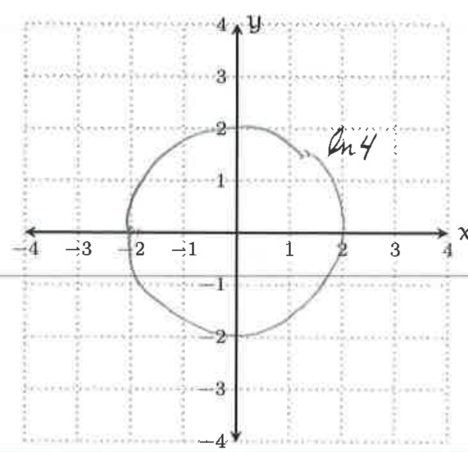
(b) State the range of f. $\boxed{\text{All real numbers}}$

(d) $f(0, \frac{1}{e}) = \ln(0^2 + (\frac{1}{e})^2) = \ln(\frac{1}{e^2}) = \boxed{-2}$

(d) Sketch the level curve for $z = \ln(4)$.

$$\ln(4) = \ln(x^2+y^2)$$
$$4 = x^2 + y^2$$
$$2^2 = x^2 + y^2$$

circle of radius 2



(e) $\nabla f(x, y) = \boxed{\left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle}$

(f) Find the rate of change of $f(x, y)$ in the direction of $\langle 5, 5\rangle$ at the point $(1, 3)$.

Direction is $\vec{u} = \frac{\langle 5,5\rangle}{|\langle 5,5\rangle|} = \frac{\langle 5,5\rangle}{\sqrt{50}} = \frac{\langle 5,5\rangle}{5\sqrt{2}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

Rate of change at (x,y) is $D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u} = \left\langle \frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

Rate of change at $(1,3)$ is thus $\left\langle \frac{2\cdot 3}{1^2+3^2}, \frac{2\cdot 1}{1^2+3^2} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle =$

$\left\langle \frac{6}{10}, \frac{2}{10} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{8}{10\sqrt{2}} = \boxed{\frac{4}{5\sqrt{2}}} = \boxed{\frac{2\sqrt{2}}{5}}$

2. (24 pts.) Evaluate each limit, if possible; if not, explain why it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$ Consider the following two approaches $(x,y)\to(0,0)$:

- $(x,y)\to(0,0)$ along x-axis $(y=0)$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = \lim_{(x,y)\to(0,0)} \frac{x}{x} = \boxed{1}$
- $(x,y)\to(0,0)$ along y-axis $(x=0)$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = \lim_{(x,y)\to(0,0)} \frac{-y}{y} = \boxed{-1}$

Since we get different values along different paths, limit $\boxed{\text{DNE}}$

(b) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy-y-2x+2}{x-1} = \lim_{(x,y)\to(1,1)} \frac{y(x-1)-2(x-1)}{x-1} = \lim_{(x,y)\to(0,0)} \frac{(x-1)(y-2)}{x-1}$

$\displaystyle = \lim_{(x,y)\to(1,1)} (y-2) = \boxed{-1}$

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3. (20 pts.) Consider the function $f(x,y) = e^{4x-x^2-y^2}$. Find all local maxima, local minima and/or saddle points.

$$\nabla f(x,y) = \langle \underbrace{e^{4x-x^2-y^2}}_{\text{positive}}(4-2x), \underbrace{-e^{4x-x^2-y^2}}_{\text{negative}} 2y \rangle = \langle 0, 0 \rangle$$

From this we see that there is one critical point $(2,0)$

$f_{xx}(x,y) = e^{4x-x^2-y^2}(4-2x)^2 + e^{4x-x^2-y^2}(-2)$

$f_{xx}(2,0) = e^4(4-2\cdot 2)^2 + e^4(-2) = -2e^4$

$f_{yy}(x,y) = e^{4x-x^2-y^2} 4y^2 - e^{4x-x^2-y^2}(2)$

$f_{yy}(2,0) = e^4 \cdot 0 + 2e^4 = -2e^4$

$f_{xy}(x,y) = e^{4x-x^2-y}(-2y)(4-2x)$

$f_{xy}(2,0) = 0$

Now, $f_{xx}(2,0) f_{yy}(2,0) - f_{xy}(2,0)^2 = (-2e^4)(-2e^4) - 0^2 = 4e^8 > 0$

Also $f_{xx}(2,0) = -2e^4 < 0$

Therefore there is a $\boxed{\text{local maximum at } (2,0)}$

4. (16 pts.) Consider $f(x, y) = \ln(xy)\tan^{-1}(x)$.

(a) $\dfrac{\partial f}{\partial x} = \dfrac{y}{xy}\tan^{-1}(x) + \ln(xy)\dfrac{1}{1+x^2} = \boxed{\dfrac{\tan^{-1}(x)}{x} + \dfrac{\ln(xy)}{1+x^2}}$

(b) $\dfrac{\partial f}{\partial y} = \dfrac{x}{xy}\tan^{-1}(x) = \boxed{\dfrac{\tan^{-1}(x)}{y}}$

(c) $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{x}{xy}\dfrac{1}{1+x^2} = \boxed{\dfrac{1}{y(1+x^2)}}$

(d) $f_x(1, 1) = \dfrac{\tan^{-1}(1)}{1} + \dfrac{\ln(1 \cdot 1)}{1+1^2} = \dfrac{\pi}{4} + \dfrac{0}{2} = \boxed{\dfrac{\pi}{4}}$

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5. (10 pts.) Sketch the domain of

$$f(x, y) = \frac{\sqrt{1-x+y}}{x+2}.$$

Need $1 - x + y \geq 0 \rightsquigarrow \{y \geq x - 1\}$

and $x + 2 \neq 0 \rightsquigarrow (x \neq -2)$

(line $x=-2$ not included)