

Name: _____

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Score: _____

Directions: Show and explain all work. Put phones away. No calculators, etc.

1. Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are three vectors in a vector space V . Without knowing any further information, is it possible to say whether or not the set $\{\mathbf{v} - \mathbf{u}, \mathbf{w} - \mathbf{v}, \mathbf{u} - \mathbf{w}\}$ is linearly independent or dependent? Explain.

Solution: Look at the equation $x(\mathbf{v} - \mathbf{u}) + y(\mathbf{w} - \mathbf{v}) + z(\mathbf{u} - \mathbf{w}) = \mathbf{0}$.

Notice that this has a non-trivial solution $x = 1, \quad y = 1, \quad z = 1$

because $1(\mathbf{v} - \mathbf{u}) + 1(\mathbf{w} - \mathbf{v}) + 1(\mathbf{u} - \mathbf{w}) = \mathbf{v} - \mathbf{u} + \mathbf{w} - \mathbf{v} + \mathbf{u} - \mathbf{w} = \mathbf{0}$.

Therefore this set is **Linearly Dependent**

2. Find a basis for the subspace $W = \{(2s - t, s, t) : s, t \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

Notice that

$$\begin{aligned} W &= \{(2s - t, s, t) : s, t \in \mathbb{R}\} \\ &= \{s(2, 1, 0) + t(-1, 0, 1) : s, t \in \mathbb{R}\} \\ &= \text{Span}\left((2, 1, 0), (-1, 0, 1)\right) \end{aligned}$$

Thus the set $S = \{(2, 1, 0), (-1, 0, 1)\}$ spans W . The set S is also linearly independent because it contains just two elements, and neither is a multiple of the other.

It follows that the set $S = \{(2, 1, 0), (-1, 0, 1)\}$ is a basis for W

3. Is the set $S = \left\{ \begin{bmatrix} 3 & 5 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \right\} \subseteq M_{2,2}$ linearly independent or dependent?

We need to find whether or not the following equation has any non-trivial solutions.

$$x \begin{bmatrix} 3 & 5 \\ 2 & -2 \end{bmatrix} + y \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} + z \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3x - y + z & 5x + y \\ 2x + 2y + 5z & -2x + y + 2z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We get the following system:

$$\begin{cases} 3x - y + z = 0 \\ 5x + y = 0 \\ 2x + 2y + 5z = 0 \\ -2x + y + z = 0 \end{cases}$$

$$\begin{array}{ccc} \left[\begin{array}{ccc|c} 3 & -1 & 1 & 0 \\ 5 & 1 & 0 & 0 \\ 2 & 2 & 5 & 0 \\ -2 & 1 & 2 & 0 \end{array} \right] & \begin{array}{c} R_1 + R_4 \rightarrow R_1 \\ \longrightarrow \end{array} & \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 5 & 1 & 0 & 0 \\ 2 & 2 & 5 & 0 \\ -2 & 1 & 2 & 0 \end{array} \right] \\ \\ \begin{array}{c} R_2 - 5R_1 \rightarrow R_2 \\ R_3 - 2R_1 \rightarrow R_3 \\ R_4 + R_1 \rightarrow R_4 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -15 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & 8 & 0 \end{array} \right] & \begin{array}{c} R_3 - 2R_2 \rightarrow R_3 \\ R_4 - R_2 \rightarrow R_4 \\ \longrightarrow \end{array} & \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -15 & 0 \\ 0 & 0 & 29 & 0 \\ 0 & 0 & 23 & 0 \end{array} \right] \\ \\ \begin{array}{c} \frac{1}{29}R_3 \rightarrow R_3 \\ \frac{1}{23}R_4 \rightarrow R_4 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -15 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] & \begin{array}{c} R_1 - 3R_3 \rightarrow R_1 \\ R_2 + 15R_3 \rightarrow R_2 \\ R_4 - R_3 \rightarrow R_4 \\ \longrightarrow \end{array} & \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

This yields the equivalent system $\begin{cases} x & = & 0 \\ y & = & 0 \\ z & = & 0 \end{cases}$

Because we get only the trivial solution, **the set S is linearly independent**