Chapter 15: Rings

2. Let R be the ring of 2×2 matrices of form $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$, that is $R = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) : a, b \in \mathbb{R} \right\}$. Does R have an identity? Such an identity would have to be of form $\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right)$ for some real numbers x and y. But then

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since this is impossible, we conclude that R has no identity.

Now consider the subset $S = \{ \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) : a \in \mathbb{R} \} \subseteq R$. This is certainly non-empty, and its is closed under multiplication, as follows. Given $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in S, we have $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} ab & 0 \\ 0 & 0 \end{smallmatrix}\right) \in S$. Furthermore, $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a-b & 0 \\ 0 & 0 \end{smallmatrix}\right) \in S$. Therefore Proposition 15.2 implies that S is a subring of R. Unlike R, the subring S has an identity. Notice that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in S$ and

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is a multiplicative identity for S.

3. Find all the units in the given rings. ■

- **b.** The units in $\mathbb{Z}_{12}$ are $\{1, 5, 7, 11\}$, namely the elements of $\mathbb{Z}_{12}$ that are relatively prime to 12.
- **d.** Describe the units of $\mathbb{M}_2(\mathbb{Z})$, i.e. the set of two-by-two matrices with integer entries.

  Such a matrix has form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, d are integers. It is a unit if it has an inverse. By elementary linear algebra, an inverse (if it exists) has form

  $$\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

  where $ad - bc$ is the determinant of the matrix. Notice that for the inverse to exist, the determinant be non-zero. But in addition, for the inverse to be in $\mathbb{M}_2(\mathbb{Z})$ its entries must be integers, so the fraction $\frac{1}{ad-bc}$ **must be an integer.** This can only happen if $ad - bc = \pm 1$.

The units of $\mathbb{M}_2(\mathbb{Z})$ are the 2×2 matrices with integer entries, and whose determinant is ± 1.

8. Is the ring $R = \mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ isomorphic to $S = \mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$?

These rings are not isomorphic. To see why, we will assume that there is an isomorphism $\varphi : R \to S$ and deduce a contradiction.

Thus suppose $\varphi : R \to S$ is an isomorphism. Notice that both R and S contain the multiplicative identity 1, as $1 = 1 + 0\sqrt{2}$ and $1 = 1 + 0\sqrt{3}$. For similar reasons $2 \in R$, and $\sqrt{2} \in R$, so it's meaningful to plug these into φ.

Let us compute. Using the fact that φ is an isomorphism, we get

$$[\varphi(\sqrt{2})]^2 = \varphi(\sqrt{2})\varphi(\sqrt{2}) = \varphi(\sqrt{2}\sqrt{2}) = \varphi(2) = \varphi(1+1) = \varphi(1) + \varphi(1) = 1 + 1 = 2.$$

(Note: Above we used $\varphi(1) = 1$, which follows from Problem 19c, below.)

Since $[\varphi(\sqrt{2})]^2 = 2$, it must be that $\varphi(\sqrt{2}) = \pm\sqrt{2} \in S$. By definition of S we have $\pm\sqrt{2} = a + b\sqrt{3}$ for some $a, b \in \mathbb{Q}$. Squaring both sides, we get $2 = a^2 + 3b^2 + 2ab\sqrt{3}$. Solving, we get

$$\sqrt{3} = \frac{2 - a^2 - 3b^2}{2ab} \in \mathbb{Q}$$

in contradiction to the fact that $\sqrt{3}$ is irrational. ■

17. For any element a of a ring with identity, $(-1)a = -a$.

Proof Let a be an element of a ring with identity 1. Recall (Proposition 15.2) that $0a = 0$. Thus $(1-1)a = 0a = 0$. By the distributive property, $(1-1)a = 0$ becomes $1a + (-1)a = 0$, which simplifies to $a + (-1)a = 0$. Add $-a$ to both sides and you get

$$\begin{aligned} -a + (a + (-1)a) &= -a + 0 \\ (-a + a) + (-1)a &= -a \\ 0 + (-1)a &= -a \\ (-1)a &= -a \end{aligned}$$

This completes the proof. ■

19. Suppose $\varphi : R \to S$ is a ring homomorphism. Prove the following.

(b) Show that $\varphi(0_R) = 0_S$.
Proof Note that $\varphi(0_R) = \varphi(0_R + 0_R) = \varphi(0_R) + \varphi(0_R)$. Thus we have $\varphi(0_R) = \varphi(0_R) + \varphi(0_R)$. Add $-\varphi(0_R)$ to both sides of this, and you get

$$-\varphi(0_R) + \varphi(0_R) = -\varphi(0_R) + \varphi(0_R) + \varphi(0_R),$$

which simplifies to $0_S = \varphi(0_R)$. ■

(c) If φ is surjective, then $\varphi(1_R) = 1_S$.
Proof We must show for any $b \in S$ that $\varphi(1_R)b = b$ and $b\varphi(1_R) = b$. Since φ is surjective, we have $b = \varphi(a)$ for some $a \in R$. Then $\varphi(1_R)b = \varphi(1_R)\varphi(a) = \varphi(1_R a) = \varphi(a) = b$. Likewise $b\varphi(1_R) = \varphi(a)\varphi(1_R) = \varphi(a1_R) = \varphi(a) = b$. ■

31. Suppose S is a subring of R, and suppose $1_S \in S$ is the identity in S and $1_R \in R$ is the identity in R. Is it necessarily true that $1_S = 1_R$?

No. Let $R = \mathbb{M}_2(\mathbb{R})$, so $1_R = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in R$. Let $S = \{\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) : a \in \mathbb{R}\}$ be the subring of R defined in Exercise 2, above. As noted there, we have $1_S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, so $1_R \neq 1_S$ in this case.