30. A ring R is called *Boolean* if $a^2 = a$ for every $a \in R$. Prove that every Boolean ring is commutative.

Proof. Assume that R is a Boolean ring. Given any $x \in R$, we have

$$\begin{aligned} x + x &= (x+x)^2 \\ &= (x+x)(x+x) \\ &= (x+x)x + (x+x)x \\ &= x^2 + x^2 + x^2 + x^2 \\ &= x + x + x + x \end{aligned}$$

Thus we get that the equation $2x = 4x$ holds for any x in a Boolean ring R. Adding $-2x$ to both sides of this equation produces $0 = 2x$, which we rewrite as $0 = x + x$, or $x = -x$.

This establishes an interesting fact about Boolean rings: Any element x of such a ring is its own additive inverse, that is $x = -x$ for every element of the ring.

Now take any two elements $a, b \in R$. Note that

$$\begin{aligned} a + b &= (a+b)^2 \\ &= (a+b)(a+b) \\ &= (a+b)a + (a+b)b \\ &= a^2 + ba + ab + b^2 \\ &= a + ba + ab + b \end{aligned}$$

This gives the equation $a + b = a + ba + ab = b$. Subtracting $a + b$ from both sides of this yields $0 = ba + ab$. Therefore $ab = -ba$. But, as noted in the first part of the proof, we have $-ba = ba$, so we get $ab = -ba = ba$. This shows $ab = ba$ for any $a, b \in R$, so R is commutative. ■