Algebra Solutions by Richard

Chapter 16: Polynomials

3. (d)

$$
\begin{array}{rrrrrrrrrrr}
 & & & & & & & 6x^2 & + & 6x & \\
\cline{5-11}
x^2 + x - 2 & | & & & 6x^4 & - & 2x^3 & + x^2 & - & 3x & + 1 \\
 & & & & 6x^4 & + & 6x^3 & - 5x^2 & & & \\
\cline{5-11}
 & & & & & & 6x^3 & + 6x^2 & - & 3x & \\
 & & & & & & 6x^3 & + 6x^2 & - & 2x & \\
\cline{7-11}
 & & & & & & & & - & 5x & + 1
\end{array}
$$

Answer: $\boxed{\text{In the ring } \mathbb{Z}_7 \text{ we have } 6x^4 - 2x^3 + x^2 - 3x + 1 = (x^2 + x - 2)(6x^2 + 6x) + 2x + 1.}$

5. Find all zeros of the following polynomials in the stated ring.

(b) $3x^3 - 4x^2 - x + 4$ in $\mathbb{Z}_5$. Since $-4 = 1$ and $-1 = 4$ for this ring, we can simplify the polynomial as $3x^3 + x^2 + 4x + 4$. The ring $\mathbb{Z}_5$ is small enough that we can just plug in its five elements individually and see what we get.

Try $x = 0$: $3 \cdot 0^3 + 0^2 + 4 \cdot 0 + 4 = 4 \neq 0$.
Try $x = 1$: $3 \cdot 1^3 + 1^2 + 4 \cdot 1 + 4 = 12 \neq 0$.
Try $x = 2$: $3 \cdot 2^3 + 2^2 + 4 \cdot 2 + 4 = 40 = 0$.
Try $x = 3$: $3 \cdot 3^3 + 3^2 + 4 \cdot 3 + 4 = 106 \neq 0$.
Try $x = 4$: $3 \cdot 4^3 + 4^2 + 4 \cdot 4 + 4 = 228 \neq 0$.

$\boxed{\text{Thus } x = 2 \text{ is the only zero of this polynomial.}}$

(d) $3x^3 + x + 1$ in $\mathbb{Z}_2$.

Whether you evaluate the polynomial for $x = 0$ or $x = 1$, the answer is 1.
Therefore $\boxed{\text{this polynomial has no zeros over the given field.}}$

9 Find all irreducible polynomials of degree 2 and 3 in $\mathbb{Z}_2[x]$.

Reason as follows: An irreducible polynomial f must have constant term 1, for otherwise $f(0) = 0$ and f has a factor of $(x - 0)$. Also, if f is irreducible over $\mathbb{Z}_2[x]$, it must have an odd number of (non-zero) terms, otherwise $f(1) = 0$, and f has a factor of $(x - 1)$. This reasoning leaves the following polynomials.

$x^2 + x + 1$
$x^3 + x^2 + 1$
$x^3 + x + 1$

23 Show that $x^p - x$ has p distinct zeros in $\mathbb{Z}_p$, for any prime p.

Proof. Recall that $\mathbb{Z}_p$ is a field for any prime p. Thus each one of its $p - 1$ non-zero elements is a unit. Then $U(p) = \{1, 2, 3, \ldots, p - 1\}$ is a group with $p - 1$ elements. By Lagrange's Theorem, we have $x^{p-1} = 1$ for each $x \in U(p)$. In other words, $x^{p-1} - 1 = 0$ for each non-zero element of $\mathbb{Z}_p$. But then $x(x^{p-1} - 1) = 0$ for all elements of $\mathbb{Z}_p$, *including* 0. (Because the first factor x is zero when $x = 0$, the second factor $x^{p-1} - 1$ is zero for all other values of x.) In other words, $x^p - x = 0$ for all p elements of $\mathbb{Z}_p$, which is just another way of saying that $x^p - x$ has p distinct zeros. ■

As $x^p - x$ has zeros $0, 1, 2, \ldots, p - 1$, it follows that $x^p - x = (x - 0)(x - 1)(x - 2)(x - 3) \cdots (x - (p - 1))$.