Chapter 12: Structure of Groups

2. List all abelian groups of order 200, up to isomorphism. (That is, no two groups on your list should be isomorphic; and for any abelian group of order 200, your list must contain it or a group isomorphic to it.)

Solution: $200 = 2^3 \cdot 5^2$. Thus according to the structure theorem of finite abelian groups, the possibilities are:

(1) $\mathbb{Z}_8 \times \mathbb{Z}_{25}$

(2) $\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_5$

(3) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{25}$

(4) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_5$

(5) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$

(6) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$

6. If n divides the order of a finite abelian group G, then G has a subgroup of order n.

Proof First we will prove that this is true if $G = \mathbb{Z}_{p^a}$, where p is prime and a is a positive integer. Suppose n divides $|\mathbb{Z}_{p^a}| = p^a$. Then n must be one of the values $p^0, p^1, p^2, p^3, p^4, \ldots, p^a$. Just note the following:

$$\begin{array}{rcllll}
\langle\ p^a\ \rangle & = & \{0\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \subseteq & \mathbb{Z}_{p^a} & \text{has order } p^0. \\
\langle\ p^{a-1}\ \rangle & = & \{p^{a-1},\ 2p^{a-1},\ 3p^{a-1},\ 4p^{a-1},\ \ldots\ ,\ p^1p^{a-1} = 0\} & \subseteq & \mathbb{Z}_{p^a} & \text{has order } p^1. \\
\langle\ p^{a-2}\ \rangle & = & \{p^{a-2},\ 2p^{a-2},\ 3p^{a-2},\ 4p^{a-2},\ \ldots\ ,\ p^2p^{a-2} = 0\} & \subseteq & \mathbb{Z}_{p^a} & \text{has order } p^2. \\
\langle\ p^{a-3}\ \rangle & = & \{p^{a-3},\ 2p^{a-3},\ 3p^{a-3},\ 4p^{a-3},\ \ldots\ ,\ p^3p^{a-3} = 0\} & \subseteq & \mathbb{Z}_{p^a} & \text{has order } p^3. \\
\langle\ p^{a-4}\ \rangle & = & \{p^{a-4},\ 2p^{a-4},\ 3p^{a-4},\ 4p^{a-4},\ \ldots\ ,\ p^4p^{a-4} = 0\} & \subseteq & \mathbb{Z}_{p^a} & \text{has order } p^4. \\
\vdots & \vdots & & \vdots & \vdots & \quad\vdots \\
\langle\ p^0 \rangle & = & \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ \ldots\ ,\ \ p^a = 0\} & = & \mathbb{Z}_{p^a} & \text{has order } p^a.
\end{array}$$

Therefore, if n divides the order of $\mathbb{Z}_{p^a}$, where p is prime, then $\mathbb{Z}_{p^a}$ has a subgroup of order n. In fact, the above reasoning shows that $\mathbb{Z}_{p^a}$ has a (cyclic) subgroup $\langle p^{a-b}\rangle$ isomorphic to $\mathbb{Z}_{p^b}$ for any $0 \leq b \leq a$.

Now let G be an arbitrary finite abelian group. By the Structure Theorem for Finite Abelian Groups,

$$G \ \cong\ \mathbb{Z}_{p_1^{a_1}} \times \mathbb{Z}_{p_2^{a_2}} \times \mathbb{Z}_{p_2^{a_2}} \times \cdots \times \mathbb{Z}_{p_n^{a_n}},$$

where the p_i are prime numbers, and $|G| = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_n^{a_n}$.

Now, if n divides $|G| = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_n^{a_n}$, then by the Fundamental Theorem of Arithmetic we can write $n = p_1^{b_1} p_2^{b_2} p_3^{b_3} \cdots p_n^{b_n}$, for some $0 \leq b_i \leq a_i$ for each index i. By the first paragraph of the proof, we know that for each index i the group $\mathbb{Z}_{p_i^{a_i}}$ has a subgroup H_i of order $p_i^{b_i}$. Then

$$H_1 \times H_2 \times \cdots \times H_n\ \subseteq\ \mathbb{Z}_{p_1^{a_1}} \times \mathbb{Z}_{p_2^{a_2}} \times \mathbb{Z}_{p_2^{a_2}} \times \cdots \times \mathbb{Z}_{p_n^{a_n}}\ \cong\ G$$

is a subgroup of order $p_1^{b_1} p_2^{b_2} p_3^{b_3} \cdots p_n^{b_n} = n$. ■

Editorial Comment: In general, if a finite group G is **not** abelian, then there may be divisors n of $|G|$ for which G has no subgroup of order n. Consider the non-abelian group A_4, which has order 12. The integer $n = 6$ divides 12, but Corollary 6.10 states that no subgroup of A_4 has order 6.