*Algebra Solutions by Richard*

# Chapter 5: Permutation Groups

**1.** Write the following permutations in cycle notation.

**(b)** $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix} = (14)(35)$

**(d)** $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix} = (42)$

**2.** Compute each of the following.

**(d)** $(1423)(34)(56)(1324) = (12)(56)$

**(i)** Here we will use the facts that $(1254)^{-2} = \left((1254)^{-1}\right)^2$ and $(1254)^{-1} = (4521)$.

$$\begin{aligned} (123)(45)(1254)^{-2} &= (123)(45)\left((1254)^{-1}\right)^2 \\ &= (123)(45)(4521)^2 \\ &= (123)(45)(4521)(4521) \\ &= (143)(25) \end{aligned}$$

**(j)** $(1254)^{100} = \left((1254)^4\right)^{25} = (1)^{25} = (1)$

**(k)** $|(1254)| = 4$

**3.** Write the following permutations as products of transpositions and identify them as even or odd.

**(b)** $(156)(234) = (15)(56)(23)(42)$ .................................. This permutation is even.

**(d)** $(142637) = (13)(12)(37)(26)(14)$ ................................ This permutation is odd.

**10.** Find an element of largest order in $S_n$ for $n = 3, 4, 5, 6, 7, 8, 9, 10$.

Recall that every permutation is a product of disjoint cycles. Also, the order of a product of disjoint cycles is the least common multiple of the orders of the cycles. For example the permutation $(123)(4567)(89)$ has order $\text{lcm}(3, 4, 2) = 12$. Thus we can find an element of largest order in $S_n$ by looking for a product of disjoint cycles, such that the lcm of the orders of the cycles is as large as possible.

**(a)** An element of largest order in $S_3$ is $(123)$, and it has order 3.

**(b)** An element of largest order in $S_4$ is $(1234)$, and it has order 4.

**(c)** An element of largest order in $S_5$ is $(12)(345)$, and it has order $2 \cdot 3 = 6$.

**(d)** An element of largest order in $S_6$ is $(123456)$, and it has order 6.

**(e)** An element of largest order in $S_7$ is $(123)(4567)$, and it has order $3 \cdot 4 = 12$.

**(f)** An element of largest order in $S_8$ is $(123)(45678)$, and it has order $3 \cdot 5 = 15$.

**(g)** An element of largest order in $S_9$ is $(1234)(56789)$, and it has order $4 \cdot 5 = 20$.

**(h)** An element of largest order in $S_{10}$ is $(01)(234)(56789)$, and it has order $2 \cdot 3 \cdot 5 = 30$.

**23.** If $\sigma$ is a cycle of odd length, then $\sigma^2$ is also a cycle.

**Proof.** Suppose $\sigma \in S_n$ is a cycle of odd length $2k+1$. Then we have

$$\sigma = (a_0\ a_1\ a_2\ a_3\ a_4\ \ldots\ a_{2k-1}\ a_{2k}),$$

for some subset $X = \{a_0, a_1, a_2, \ldots, a_{2k}\} \subseteq \{1, 2, 3, 4, \ldots, n\}$. This means that

$$\sigma(a_i) = a_{i+1}$$

for each $a_i \in X$ (where the $i+1$ is computed mod $2k+1$), while $\sigma(x) = x$ for each $x \in \{1, 2, 3, \ldots, n\} \setminus X$.

Now consider the effect of $\sigma^2 = \sigma\sigma$. The above paragraph implies that for each $a_i \in X$ we have $\sigma^2(a_i) = \sigma(\sigma(a_i)) = \sigma(a_{i+1}) = a_{i+2}$, where, again, the addition $i+2$ is done modulo $2k+1$ and $\sigma^2(x) = x$ for any $x$ not in $X$. From this we see that

$$\sigma^2 = (a_0\ a_2\ a_4\ a_6\ \ldots\ a_{2k},\ \ a_1,\ a_3,\ a_5,\ \ldots\ a_{2k-1}),$$

and is therefore a cycle. ■

**31.** For $\alpha, \beta \in S_n$, define $\alpha \sim \beta$ if there is some $\sigma \in S_n$ for which $\sigma\alpha\sigma^{-1} = \beta$. Show that $\sim$ is an equivalence relation.

**Proof.** We need to show that $\sim$ is reflexive, symmetric and transitive.

(1) Observe that for any $\alpha \in S_n$ we have $\sigma\alpha\sigma^{-1} = \alpha$, where $\sigma = \text{id} \in S_n$. Thus $\sigma$ is reflexive.

(2) Suppose $\alpha \sim \beta$. By definition of $\sim$ this means that there is some $\sigma \in S_n$ for which $\sigma\alpha\sigma^{-1} = \beta$. Left-multiplying both sides of this by $\sigma^{-1}$ gives us $\alpha\sigma^{-1} = \sigma^{-1}\beta$. Now right-multiplying both sides by $\sigma$ gives $\alpha = \sigma^{-1}\beta\sigma$. Rewrite this as

$$\sigma^{-1}\beta(\sigma^{-1})^{-1} = \alpha. \tag{1}$$

As $\sigma \in S_n$, we also have $\sigma^{-1} \in S_n$, so Equation (1) and the definition of $\sim$ now yields $\beta \sim \alpha$. We've shown $\alpha \sim \beta$ implies $\beta \sim \alpha$, so $\sim$ is symmetric.

(3) Now we will show that $\sim$ is transitive. Suppose $\alpha \sim \beta$ and $\beta \sim \gamma$. Then there are permutations $\sigma, \tau \in S_n$ for which $\sigma\alpha\sigma^{-1} = \beta$ and $\tau\beta\tau^{-1} = \gamma$. Inserting the value for $\beta$ given by the first equation into the second and using associativity yields

$$\tau(\sigma\alpha\sigma^{-1})\tau^{-1} = \gamma$$
$$(\tau\sigma)\alpha(\sigma^{-1}\tau^{-1}) = \gamma$$
$$(\tau\sigma)\alpha(\tau\sigma)^{-1} = \gamma.$$

As $\tau, \sigma \in S_n$, we also have $\tau\sigma \in S_n$. Hence the above equation shows $\alpha \sim \gamma$.
Having shown that $\alpha \sim \beta$ and $\beta \sim \gamma$ together imply $\alpha \sim \gamma$, we see that $\sim$ is transitive.

The above considerations show that $\sim$ is an equivalence relation. ■

**34.** Suppose $\alpha$ is an even permutation. Prove that $\alpha^{-1}$ is also even. Is the same true if $\alpha$ is odd?

**Proof.** Suppose $\alpha$ is even. This means there is an even integer $p$ and transpositions $\tau_1, \tau_2, \ldots \tau_p$ for which $\alpha = \tau_1\tau_2\tau_3 \cdots \tau_{p-1}\tau_p$. Now the inverse of each transposition $\tau_i$ is of course $\tau_i^{-1} = \tau_i$. Therefore we have

$$\begin{aligned} \alpha^{-1} &= (\tau_1\tau_2\tau_3 \cdots \tau_{p-1}\tau_p)^{-1} \\ &= \tau_p^{-1}\tau_{p-1}^{-1} \cdots \tau_3^{-1}\tau_2^{-1}\tau_1^{-1} \\ &= \tau_p\tau_{p-1} \cdots \tau_3\tau_2\tau_1. \end{aligned}$$

This expresses $\alpha^{-1}$ as a product of an even number of transpositions, so $\alpha^{-1}$ is even.

By the same reasoning, if $\alpha$ is odd, its inverse $\alpha^{-1}$ will be odd too. Thus it is not true that if $\alpha$ is odd, then its inverse is even. ■