Algebra Solutions by Richard

Chapter 9: Isomorphisms

2 Let H be the set of matrices of form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{R})$. Prove that $\mathbb{C}^*$ is isomorphic to H.

Let's begin by carefully writing down this set of matrices in set notation. Notice that the determinant of such a matrix is $a^2 + b^2$, so H is the set $H = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R},\ a^2 + b^2 \neq 0 \right\}$.

Now, define a map $\varphi : \mathbb{C}^* \to H$ as

$$\varphi(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

This is a well-defined map, because if $a + bi \in \mathbb{C}^*$, then $0 < |a + bi| = \sqrt{a^2 + b^2}$, so $a^2 + b^2 \neq 0$, hence $\varphi(a + bi)$ really is an element of H. This map is clearly surjective, for any matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ in H is the image of the complex number $a + bi$ in $\mathbb{C}^*$. Also, the map is injective, because if $\varphi(a + bi) = \varphi(c + di)$ we get

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} c & d \\ -d & c \end{pmatrix},$$

and hence $a = c$ and $b = d$, so $a + bi = c + di$. As it is surjective and injective, φ is a bijection.

To complete the proof we need to show $\varphi((a + bi)(c + di)) = \varphi(a + bi)\varphi(c + di)$. Indeed

$$\begin{aligned}
\varphi(a + bi)\varphi(c + di) &= \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix} \\
&= \begin{pmatrix} ac - bd & ad + bc \\ -bc - ad & -bd + ac \end{pmatrix} \\
&= \begin{pmatrix} ac - bd & ad + bc \\ -(ad + bc) & ac - bd \end{pmatrix} \\
&= \varphi((ac - bd) + (ad + bc)i) \\
&= \varphi((a + bi)(c + di)).
\end{aligned}$$

This completes the demonstration.

5 Show $U(5)$ is isomorphic to $U(10)$, but $U(12)$ is not.

To answer this, it may be helpful to consider the Caley tables, as follows.

$U(5) = \{1, 2, 3, 4\}$

·	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$U(10) = \{1, 3, 7, 9\}$

·	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

$U(12) = \{1, 5, 7, 11\}$

·	1	5	7	11
1	1	5	7	11
5	5	1	11	1
7	7	11	1	5
11	11	7	5	1

From these, it is easy to make the following calculations:
For $U(5)$ we have $\langle 2 \rangle = \{2^k : k \in \mathbb{Z}\} = \{2, 4, 3, 1\} = U(5)$, so $U(5)$ is cyclic.

For $U(10)$ we have $\langle 3 \rangle = \{3^k : k \in \mathbb{Z}\} = \{3, 9, 7, 1\} = U(10)$, so $U(10)$ is cyclic.

Since $U(5)$ and $U(10)$ are each cyclic with four elements, Theorem 9.3 says $U(5) \cong \mathbb{Z}_4 \cong U(10)$.

Concerning $U(12)$, $\langle 1 \rangle = \{1\}$, $\langle 5 \rangle = \{1, 5\}$, $\langle 7 \rangle = \{1, 7\}$ and $\langle 11 \rangle = \{1, 11\}$. Therefore $U(12)$ is **not** cyclic, hence it cannot be isomorphic to the cyclic groups $U(5)$ and $U(10)$. (By Theorem 9.1.)

9. We have already seen (Exercise 7 in Chapter 1) that the set $\mathbb{R} \setminus \{-1\}$ is a group under the operation $a * b = a + b + ab$. Now show that this group is isomorphic to $\mathbb{R}^*$.

Proof. Consider the map $\varphi : \mathbb{R} \setminus \{-1\} \to \mathbb{R}^*$ defined as $\varphi(x) = 1 + x$.

This map is surjective because if $b \in \mathbb{R}^*$, then $b - 1 \in \mathbb{R} \setminus \{-1\}$, and $\varphi(b-1) = 1 + b - 1 = b$. This map is also injective, for if $\varphi(x) = \varphi(y)$, then $1 + x = 1 + y$, so $x = y$. As φ is both surjective and injective, it is bijective.

To show φ is an isomorphism, we now need only to verify $\varphi(x * y) = x \cdot y$ for each $x, y \in \mathbb{R} \setminus \{-1\}$. Thus take $x, y \in \mathbb{R} \setminus \{-1\}$ and observe

$$\varphi(x * x) = \varphi(x + y + xy) = 1 + x + y + xy = (1 + x)(1 + y) = \varphi(x)\varphi(y).$$

Therefore $\varphi(x * y) = \varphi(x) \cdot \varphi(y)$, so φ is an isomorphism. ■

Note: The map $\varphi : \mathbb{R} \setminus \{-1\} \to \mathbb{R}^*$ defined as $\varphi(x) = \frac{1}{1+x}$ also works.

16. Find the order of each of the following elements.

(b) $(6, 15, 4)$ in $\mathbb{Z}_{30} \times \mathbb{Z}_{45} \times \mathbb{Z}_{24}$ .... has order $\text{lcm}(5, 3, 6) = 30$.
(d) $(8, 8, 8)$ in $\mathbb{Z}_{10} \times \mathbb{Z}_{24} \times \mathbb{Z}_{80}$ .... has order $\text{lcm}(5, 3, 10) = 30$.

22. Let G be a group of order 20. If G has subgroups H and K of orders 4 and 5, respectively, such that $hk = kh$ for all $h \in H$ and $k \in K$, prove that G is the internal direct product of H and K.

Proof. According to the definition of the internal direct product, we must check three criteria:

- $G = HK = \{hk : h \in H, k \in K\}$
- $H \cap K = \{e\}$
- $hk = kh$ for all $h \in H$ and $k \in K$

The third criteria already holds. To prove that the second holds, take an arbitrary element $a \in H \cap K$. Then $a \in H$ and $a \in K$. Thus $\langle a \rangle \subseteq H$ and $\langle a \rangle \subseteq K$. By Lagrange's Theorem, $|\langle a \rangle|$ divides $|H| = 5$, and likewise $|\langle a \rangle|$ divides $|H| = 4$. Consequently $|\langle a \rangle| = 1$, that is, $\langle a \rangle = \{e\}$. It follows that $a = e$. We have now shown that any element $a \in H \cap K$ has to be $a = e$, so $H \cap K = \{e\}$.

Finally, we need to show $G = HK$. Consider the map $\varphi : H \times K \to G$ defined as $\varphi((h, k)) = hk$. We claim that this map is injective. To see this, suppose $\varphi((h, k)) = \varphi((h', k'))$, so $hk = h'k'$. From this, $h'^{-1}h = k'k^{-1}$. By closure in a subgroup. the expression on the left is in H and the expression on the right is in K. But since $H \cap K = \{e\}$, it must be that $h'^{-1}h = k'k^{-1} = e$. This yields $h = h'$ and $k = k'$, so $(h, k) = (h', k')$, which proves that φ is injective.

Now we have an injective map $\varphi : H \times K \to G$. But since $|H \times K| = 20 = |G|$, this map must also be surjective. Since it is surjective, and $g \in G$ can be written as $g = \varphi(h, k) = hk$ for some $(h, k) \in H \times K$. Consequently, $G = HK$. We have now verified all three criteria. ■

50. Prove that $A \times B$ is abelian if and only if both A and B are abelian.

Proof. Suppose $A \times B$ is abelian. Take any $x, y \in A$ and $z, w \in B$. Since $A \times B$ is abelian, we have

$$\begin{aligned} (xy, zw) &= (x, z)(y, w) \\ &= (y, w)(x, z) = (yx, wz). \end{aligned}$$

So $(xy, zw) = (yx, wz)$, which means $xy = yx$ and $zw = wz$. Hence both A and B are abelian.

Conversely suppose both A and B are abelian. Take any two $(x, z), (y, w) \in A \times B$. Since $xy = yx$ and $zw = wz$, we have $(x, z)(y, w) = (xy, zw) = (yx, wz) = (y, w)(x, z)$. Thus $A \times B$ is abelian. ■