Algebra Solutions by Richard

Chapter 10: Homomorphisms and Factor Groups

1. For each of the following groups G, determine whether H is a normal subgroup of G. If H is normal, write out a Cayley table for the factor group G/H.

(a) $G = S_4$ and $H = A_4$.

Here H consists of all of the even permutations in S_4. Now, if σ is an even permutation, then σH consists of the even permutations in H multiplied on the left by the even permutation σ, so σH is a set of even permutations. Hence $\sigma H \subseteq H$, so $\sigma H = H$ by Lemma 6.1. For the same reason $H\sigma = H$. Therefore $\sigma H = H\sigma = H$ when σ is even.

On the other hand, if σ is an odd permutation, then σH consists of the even permutations in H multiplied on the left by the odd permutation σ, so it follows that σH is the set of odd permutations. For the same reason, $H\sigma$ is the set of odd permutations in S_4, so we have $\sigma H = H\sigma$.

The above two paragraphs establish that $\sigma H = H\sigma$ for every $\sigma \in S_4$, so H is normal.

The Cayley table for the factor group is as follows.

	$(1)H$	$(12)H$
$(1)H$	$(1)H$	$(12)H$
$(12)H$	$(12)H$	$(1)H$

(b) $G = A_5$ and $H = \{(1), (123), (132)\}$.

In this example, H is **not** a normal subgroup of G. To see this, consider the permutation $(124) \in A_5$. We will show that $(124)H \neq H(124)$.

First note $(124)H = \{(124)(1), (124)(123), (124)(132)\} = \{(124), (14)(32), (134)\}$.
Next note $H(124) = \{(1)(124), (123)(124), (132)(124)\} = \{(124), (13)(24), (242)\}$.

Therefore we have established $(124)H \neq H(124)$, so H is not normal.

(e) $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$.

Since G is abelian, it automatically follows that $n + H = H + n$ for any $n \in G$, so H is normal.

The cosets are as follows.
$0 + H = \{\dots, -10, -5, 0, 5, 10, 15, \dots\}$
$1 + H = \{\dots, -9, -4, 1, 6, 11, 16, \dots\}$
$2 + H = \{\dots, -8, -3, 2, 7, 12, 17, \dots\}$
$3 + H = \{\dots, -7, -2, 3, 8, 13, 18, \dots\}$
$4 + H = \{\dots, -6 - 1, 4, 9, 14, 19, \dots\}$

The Cayley table for the factor group is as follows.

$+$	$0+H$	$1+H$	$2+H$	$3+H$	$4+H$
$0+H$	$0+H$	$1+H$	$2+H$	$3+H$	$4+H$
$1+H$	$1+H$	$2+H$	$3+H$	$4+H$	$0+H$
$2+H$	$2+H$	$3+H$	$4+H$	$0+H$	$1+H$
$3+H$	$3+H$	$4+H$	$0+H$	$1+H$	$2+H$
$4+H$	$4+H$	$0+H$	$1+H$	$2+H$	$3+H$