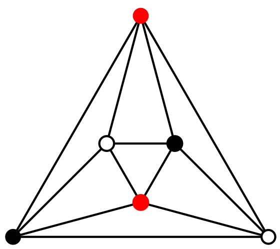
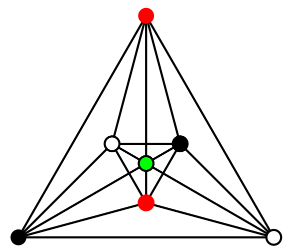
6.3.21 Determine $\nu(K_{1,2,2,2})$. Use this to compute $\nu(K_{2,2,2,2})$.

First, notice that we can draw $K_{2,2,2}$ on the plane, as shown below, left. Vertices of one partite set are colored red, those of the second are colored black, and those of the third are colored white. We can then draw $K_{1,2,2,2}$ on the plane, with three crossings, by adding a new vertex in the center (colored green) and connecting it to all six vertices of $K_{2,2,2}$. (See the drawing below, right.) From this we can tell that $\boxed{\nu(K_{1,2,2,2}) \leq 3.}$



$K_{2,2,2}$ on the plane



$K_{1,2,2,2}$ on the plane with three crossings

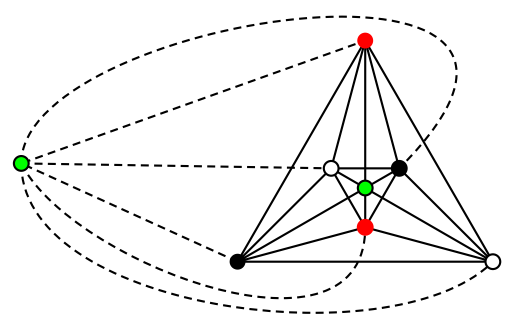
Now, $K_{1,2,2,2}$ has $v = 18$ edges and $e = 7$ vertices. By Lemma 6.1.23, it is non-planar, as

$$18 = e > 3v - 6 = 15.$$

In fact, we can see from this that we must remove at least three edges from $K_{1,2,2,2}$ before getting a planar graph. Further, we can see from the drawing above (right) that we can get a planar graph by removing precisely the three edges incident with the green vertex that cross an edge. Therefore the maximum number of edges in a planar subgraph of $K_{1,2,2,2}$ is $k = 15$. By Lemma 6.3.13, $\boxed{\nu(K_{1,2,2,2}) \geq m - k = 18 - 15 = 3.}$ Therefore $\boxed{\nu(K_{1,2,2,2}) = 3.}$

Now let's consider $\nu(K_{2,2,2,2})$. Below is a picture of it obtained by adding a new green vertex to the picture of $K_{1,2,2,2}$ above, and connecting it to all the non-green vertices. Notice that there are 6 crossings, so $\boxed{\nu(K_{2,2,2,2}) \leq 6.}$

$K_{2,2,2,2}$ on the plane with six crossings:



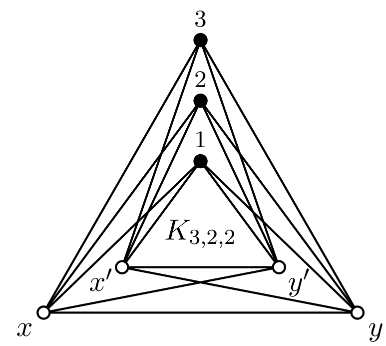
Now, $K_{2,2,2,2}$ has $v = 8$ vertices and $e = 24$ edges. Again, we apply Lemma 6.1.23 to

$$24 = e > 3v - 6 = 18,$$

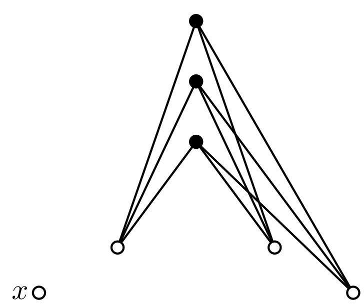
to see that least six edges must be removed from $\nu(K_{2,2,2,2})$ before obtaining a planar graph. Indeed the drawing above shows that exactly six edges can be removed to get a planar graph: the three dashed edges that cross, and the three edges from the inner green vertex that cross. Hence the maximum number of edges in a planar subgraph of $K_{2,2,2,2}$ is $24 - 6 = 18$. By Lemma 6.3.13, $\boxed{\nu(K_{2,2,2,2}) \geq m - k = 24 - 18 = 6.}$ Consequently $\boxed{\nu(K_{2,2,2,2}) = 6.}$ ■

6.3.22 Prove that $K_{3,2,2}$ has no planar subgraph with 15 edges. Deduce that $\nu(K_{3,2,2}) \geq 2$.

Below is a drawing of $K_{3,2,2}$ with the vertices in the partite set of size 3 colored black and numbered 1, 2, 3. There are two types of edges: those that join two partite sets of size 2, and those that join a partite set of size 2 to one of size 3.

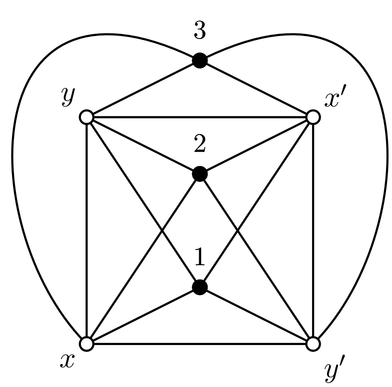


If we delete an edge joining the partite sets of size 2 we are still left with a graph that has a $K_{3,3}$ subgraph, as indicated below.



Moreover, if we delete an edge joining a partite set of size 2 to the partite set of size 3 (say an edge incident with x, above) we still are left a $K_{3,3}$ subgraph.

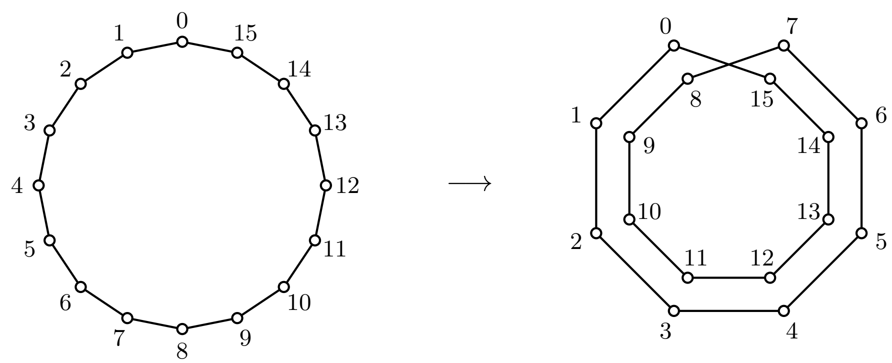
Therefore deleting a single edge from $K_{3,2,2}$ results in non-planar graph. As $K_{3,2,2}$ has $m = 16$ edges, we conclude that it has no planar subgraph with 15 edges. Therefore a planar subgraph with a maximum number of edges has $k < 15$ edges. By Lemma 6.3.13, $\boxed{\nu(K_{3,2,2}) \geq m - k = 16 - k \geq 2.}$ In fact, the drawing below shows $K_{3,2,2}$ with exactly two crossings, so $\boxed{\nu(K_{3,2,2}) = 2.}$



6.3.23 Let M_n be the graph obtained from the n-cycle by adding an edge from each vertex to its opposite vertex (if n is even) or its two near-opposite vertices (if n is odd). Find $\nu(M_n)$.

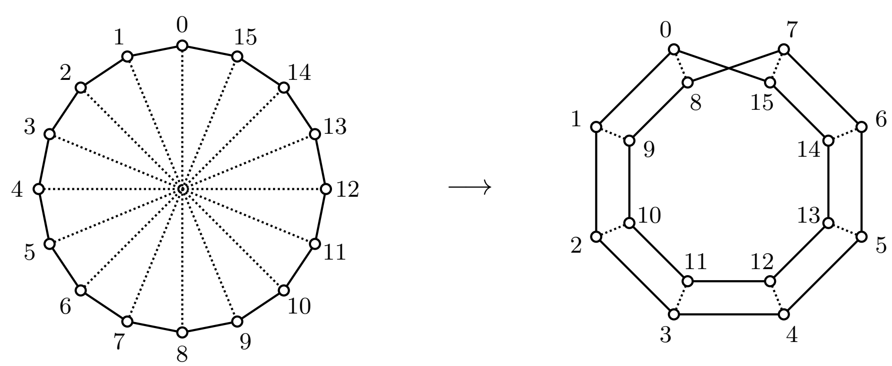
Notice that $M_3 = K_3$, $M_4 = K_4$, $M_5 = K_5$ and $M_6 = K_{3,3}$. Thus M_5 and M_6 are non-planar. Further, M_n is also non-planar for all $n > 6$ because M_n has $K_{3,3}$ as a subgraph. Therefore $\nu(M_n) = 0$ for $n \leq 4$ and $\nu(M_n) \geq 1$ whenever $n > 4$. In fact we will show that $\nu(M_n) = 1$ for all $n > 4$ by drawing M_n with just one crossing.

The trick is to double the cycle C_n back on itself, so that pairs of opposite vertices are next to each other, as shown below.



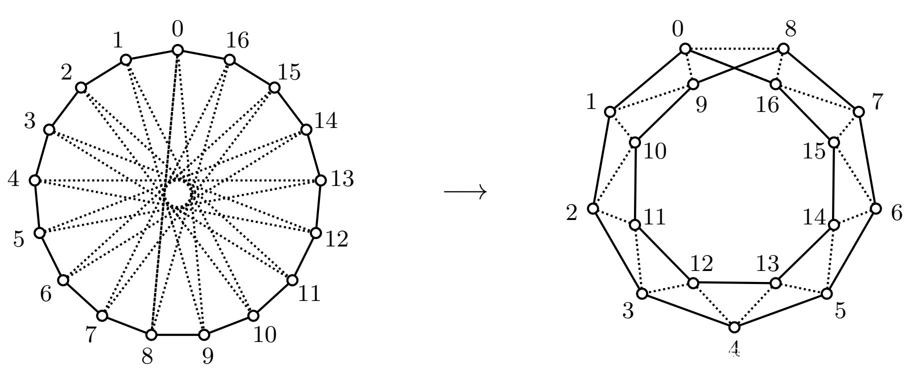
Below are two pictures of M_{16}. The standard view is on the left, with the diagonal edges shown dotted. A drawing with just one crossing is on the right. We can carry out this construction for any even n, so $\nu(M_n) = 1$ when n is even and greater than five.

M_{16}



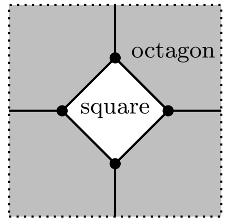
Below are two pictures of M_{17}. The standard view is on the left, with near-diagonal edges shown dotted. A drawing with just one crossing is on the right. We can carry out this construction for any odd n, so $\nu(M_n) = 1$ when n is odd and greater than four.

M_{17}



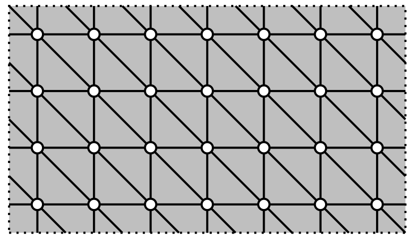
6.3.32 Construct a 3-regular simple non-bipartite graph on the torus with every face of even length.

The graph K_4 meets the description, and here it is on the torus with two faces, one a square and the other an octagon.



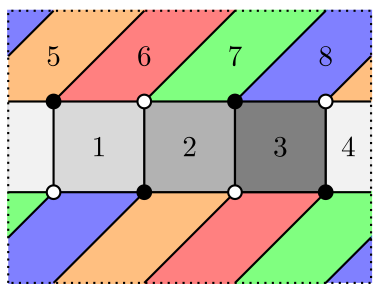
6.3.33 Let $n \geq 9$ be neither prime nor twice a prime. Construct a 6-regular n-vertex toroidal graph.

Given the conditions on n, we have $n = ab$ for integers $a, b \geq 3$. Place an $a \times b$ square grid on the torus, as shown, and then add the indicated diagonals. The result is a 6-regular n-vertex graph on the torus.

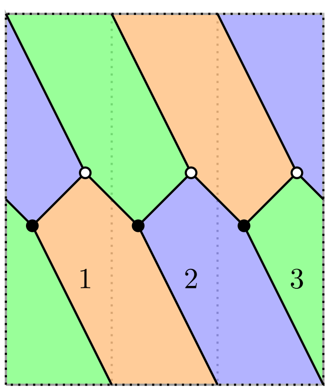


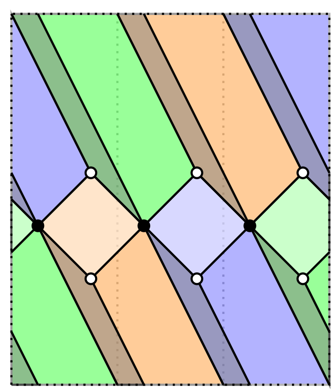
6.3.34 Construct regular embeddings of $K_{4,4}$, $K_{3,3}$ and $K_{3,6}$ on the torus. (Regular means that all faces have the same length.)

First, here is $K_{4,4}$ embedded on the torus with 8 square faces. Vertices of one partite set are colored black and those in the other partite set are colored white.



Next, below (left) is an embedding of $K_{3,3}$ on the torus with three hexagon faces. Vertices of one partite set are colored white and vertices of the other partite set are colored black.





Notice that each hexagon face of the embedding of $K_{3,3}$ contains all six vertices. So to embed $K_{3,6}$ on the torus, we can take our embedding of $K_{3,3}$ (on the left) and add a new white vertex inside each of the three faces. Then we connect each new white vertex to the three black vertices on the face that it is in. The result is and embedding of $K_{3,6}$ with 9 square faces (right). ■

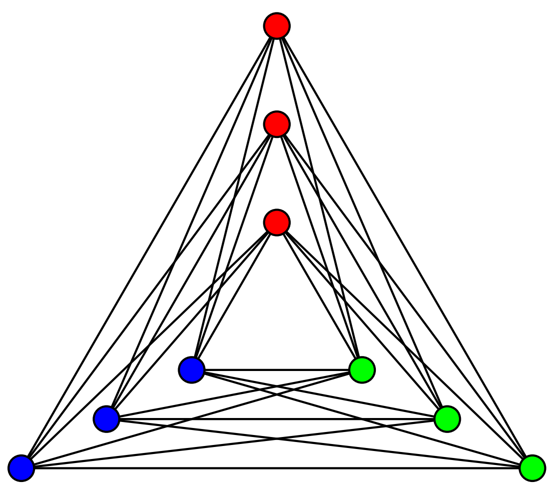
6.3.36 Find a lower bound on $\gamma(K_{3,3,n})$. Use it to determine $\gamma(K_{3,3,n})$ exactly for $n \leq 3$.

Notice that $K_{3,3,n}$ has $v = 6 + n$ vertices and $e = 6n + 9$ edges. Suppose $K_{3,3,n}$ is 2-cell embedded in some surface S_γ. Using Lemma 6.3.24, we know that

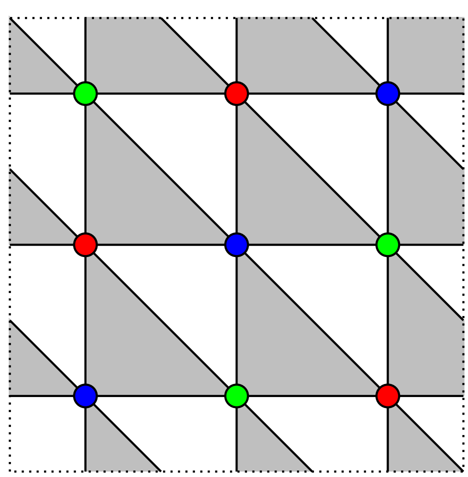
$$\begin{aligned} e &\leq 3(v - 2 + 2\gamma) \\ 6n + 9 &\leq 3\big((6+n) - 2 + 2\gamma\big) \\ 3n - 3 &\leq 6\gamma \\ \frac{n-1}{2} &\leq \gamma. \end{aligned}$$

Therefore our lower bound is $\boxed{\frac{n-1}{2} \leq \gamma(K_{3,3,n}).}$

This yields $0 \leq \gamma(K_{3,3,1})$, $1 \leq \gamma(K_{3,3,2})$, and $1 \leq \gamma(K_{3,3,1})$. Actually, this is not really all that exciting, because we know that each of these graphs contains a $K_{3,3}$ and is therefore non-planar; hence its genus is at least 1. In what follows we will show that each has genus exactly 1 by embedding it in the torus. First, consider $K_{3,3,3}$, drawn below with partite sets colored red, blue and green, respectively.

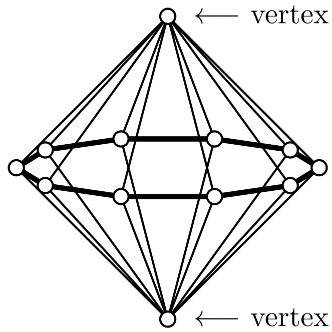


Below is the same graph $K_{3,3,3}$ drawn on the torus with 18 triangular faces. Thus $\gamma(K_{3,3,3}) = 1$. Since $K_{3,3,2}$ and $K_{3,3,1}$ are both non-planar, and both are subgraphs of the genus-1 graph $K_{3,3,3}$, it follows that $\gamma(K_{3,3,2}) = 1$ and $\gamma(K_{3,3,1}) = 1$ also.

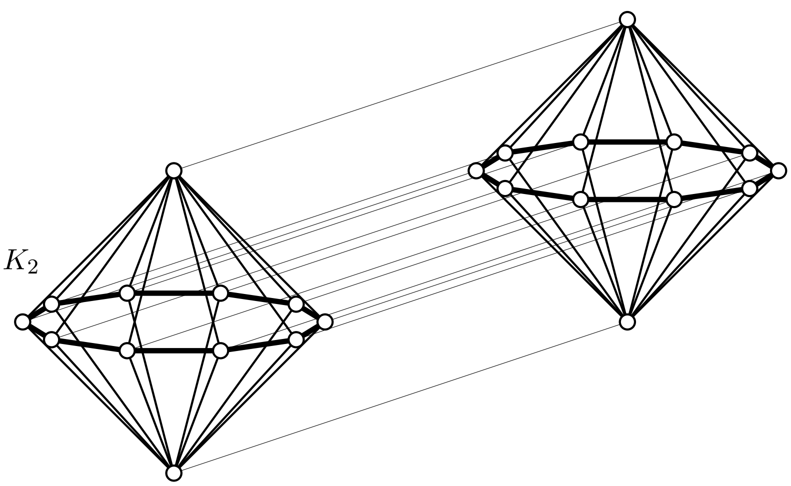


6.3.37 Prove that for every positive integer k, there exists a planar graph G for which $\gamma(G \square K_2) \geq k$.

Given a positive integer $n > 5$, let G be the graph obtained from the cycle C_{n-2} by adding two new vertices and joining them to all the vertices of C_{n-2}. You can think of this as if C_{n-2} is on the x, y-plane, and the two additional vertices are above and below it, on the y-axis. Then when we add the new edges we get a double cone. Clearly the graph G is planar (embedded on a sphere) with $2(n-2)$ triangular faces. Note that $n(G) = n$, and G is maximal planar, with $3(n-2) = 3n - 6$ edges.



Take the product $G \square K_2$. This product is drawn below. Notice that $G \square K_2$ consists of two copies of G, plus $n(G)$ new edges joining corresponding pairs of vertices from the two copies of G. Thus $n(G \square K_2) = 2n$ and $e(G \square K_2) = 2e(G) + n(G) = 2(3n-6) + n = 7n - 12$.



Suppose that $G_n \square K_2$ is 2-cell embedded in some surface S_γ. We can get a lower bound on γ as follows. By Lemma 6.3.24, we know that

$$e(G \square K_2) \leq 3(n(G \square K_2) - 2 + 2\gamma).$$

Thus

$$\begin{aligned} 7n - 12 &\leq 3(2n - 2 + 2\gamma) \\ n - 6 &\leq 2\gamma \\ \frac{n-6}{2} &\leq \gamma. \end{aligned}$$

Thus $\frac{n-6}{2} \leq \gamma(G \square K_2)$. Given any positive integer k, we can construct G as above, with n big enough so that $k < \frac{n-6}{2}$. Then $k \leq \gamma(G \square K_2)$. ∎