Test One | Advanced Graph Theory | February 26, 2019

MATH 656

Name: ______________________ R. Hammack Score: ________

Directions: Choose any four questions. Each of your four chosen questions is 25 points, for a total of 100 points. If you do more than four questions, please clearly indicate which of the four you want to contribute toward your 100 points.

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1. Say G is a simple graph with 19 edges, and $\delta(G) \geq 3$. Knowing nothing else about G, answer the following questions.
   (a) What is the maximum number of vertices that G could have?
   (b) What is the maximum number of vertices that G could have for which we can be 100% certain that G is non-planar?

2. Suppose D is an n-vertex simple digraph with no cycles. Prove that the vertices of G can be ordered as $v_1, v_2, \ldots, v_n$ such that if $v_i v_j \in E(D)$, then $i < j$.

3. Let $n \in \mathbb{N}$. Prove that there is an n-vertex tournament in which every vertex is a king if and only if $n \notin \{2, 4\}$.

4. Prove that no tournament has exactly two kings.

5. Prove that every simple planar graph with at least four vertices has at least four vertices of degree less than 6.

6. Prove that if G is planar and every face in a plane embedding of G has even length, then G is bipartite.

7. Let G be the graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \big\{v_i v_j : 1 \leq \big|i - j\big| \leq 3\big\}$. Prove that G is maximal planar.

8. Let G be the graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \big\{v_i v_j : 1 \leq \big|i - j\big| \leq 4\big\}$. Prove that $\nu(G) = n - 4$. (You may use problem 7 even if you didn't do it.)

9. Suppose n is a fixed odd integer. Prove that in all drawings of K_n, the parity of the number of crossings is the same.

10. Suppose G has v vertices, e edges, and its girth is g. Prove that $\gamma(G) \geq \dfrac{e}{2}\left(1 - \dfrac{2}{g}\right) - \dfrac{v}{2} + 1$.