Test One | Advanced Graph Theory | February 26, 2019

MATH 656

Name: ____________________ R. Hammack Score: ______

Directions: Choose any four questions. Each of your four chosen questions is 25 points, for a total of 100 points. If you do more than four questions, please clearly indicate which of the four you want to contribute toward your 100 points.

Here are solutions for 9 of the 10 questions on the test. (I only wrote solutions to problems attempted on the actual tests.) To make the solutions fit neatly on the pages, they are not exactly in numerical order. –Richard H.

1. Say G is a simple graph with 19 edges, and $\delta(G) \geq 3$. Knowing nothing else about G, answer the following questions.

   (a) What is the maximum number of vertices that G could have?

   No vertex of G has degree lower than 3, and the sum of its vertex degrees is $2e = 2 \cdot 19 = 38$. We will have potentially the most vertices if the number of vertices of lowest degree 3 is maximized. Under the constraints, this can be done with eleven vertices of degree 3 and one of degree 5, which would be a graph with the following degree sequence.

$$5, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3 \tag{1}$$

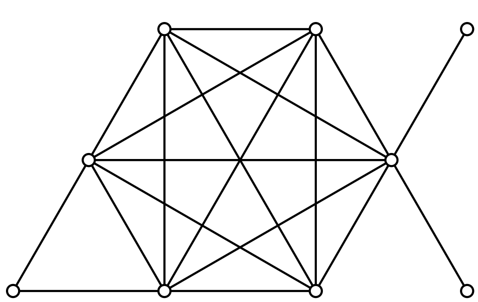
   Such a graph has 12 vertices. The question is now whether or not this degree sequence is graphic; if it is then the answer to our question is that that G can have at most 12 vertices. Let's test the above sequence with Havel-Hakimi.

$$\begin{array}{cccccccccccc}
5 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\
 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 \\
 & & 2 & 2 & 2 & 3 & 3 & 2 & 2 & 2 & 2 & 2 \\
 & & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
 & & & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
 & & & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\
 & & & & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 \\
 & & & & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
 & & & & & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\
 & & & & & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & 0 & 0 & 1 & 1 & 1 & 1 \\
 & & & & & & 1 & 1 & 1 & 1 & 0 & 0 \\
 & & & & & & & 0 & 1 & 1 & 0 & 0 \\
 & & & & & & & 1 & 1 & 0 & 0 & 0 \\
 & & & & & & & & 0 & 0 & 0 & 0
\end{array}$$

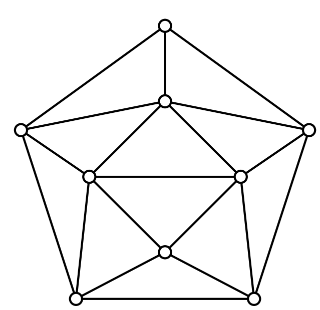
   This final sequence is indeed graphical (four vertices), so the original sequence (1) is graphical. Therefore there is a graph G with 19 edges and 12 vertices, 11 of which are of degree 3. Thus the answer to our question is that G **could have up to 12 vertices, but no more that that.** ■

   (b) What is the maximum number of vertices that G could have for which we can be 100% certain that G is non-planar?

   We will know G is non-planar provided that $e > 3v - 6$, or $19 > 3v - 6$, meaning $\frac{25}{3} > v$. Thus if G has 8 or fewer vertices we know G is non-planar. But if G had more than 8 vertices we can't be sure whether or not it's planar. It could have a Kuratowski subgraph (below left), or be planar (below right). ■



Non-planar, 9 vertices, 19 edges



Planar, 9 vertices, 19 edges

2. Suppose D is an n-vertex simple digraph with no cycles. Prove that the vertices of D can be ordered as $v_1, v_2, \ldots, v_n$ such that if $v_iv_j \in E(D)$, then $i < j$.

   Proof: We use induction on the order n of D. The statement is clearly true for $n = 1$ and $n = 2$.

   Now let $n > 2$ and assume that for every digraph D' with fewer than n vertices and no cycles, its vertices can be ordered as $v_1, v_2, \ldots, v_{n'}$ such that if $v_iv_j \in E(D')$, then $i < j$.

   Suppose D is a digraph with n vertices and no cycles. We need to prove that the vertices of D can be ordered as $v_1, v_2, \ldots, v_n$ such that if $v_iv_j \in E(D)$, then $i < j$.

   First we claim that D has a vertex with zero out-degree. Start at an arbitrary vertex x of D. If x has zero out-degree, then we are done. Otherwise we can follow a directed path: Beginning at x, move along an arc xx_1 to a vertex x_1. If x_1 has zero out-degree, then stop at x_1. Otherwise there is an arc x_1x_2, so follow it to x_2. If x_2 has zero out-degree, then stop at x_2. Otherwise there is an arc x_2x_3, so follow it to x_3. Continuing this process we get a directed path $xx_1x_2x_3\ldots$ in D. Since D has no cycles, this path never meets itself, and since D is finite, this path cannot continue forever. Thus it must terminate at some vertex x_m, which necessarily has out-degree zero. Our claim is proved.

   Take a vertex of D that has zero out-degree, and label it v_n. Let $D' = D - v_0$, so D' has no cycles (since D had none) and D' has fewer than n vertices. By the inductive hypothesis, we can label the vertices of D' as $v_1, v_2, \ldots, v_{n-1}$ such that if $v_iv_j \in E(D')$, then $i < j$. Putting v_n back in the mix, we get a labeling $v_1, v_2, \ldots, v_{n-1}, v_n$ of the vertices of D.

   Take any arc v_iv_j of D. If $v_iv_j \in E(D')$, then $i < j$ by the previous paragraph. On the other hand, if $v_iv_j \notin E(D')$, then one of its endpoints is v_n. And since v_n has out-degree zero, it must be that $j = n$, that is, $v_iv_j = v_iv_n$. In this case certainly we have $i < j = n$.

   In summary, we have labeled the vertices of Das $v_1, v_2, \ldots, v_n$ such that if $v_iv_j \in E(D)$, then $i < j$. ■

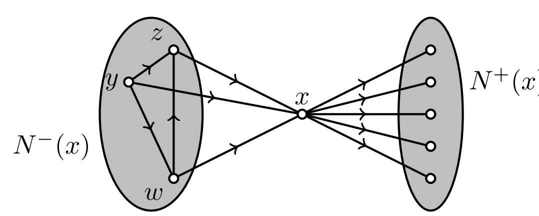
4 Prove that no tournament has exactly two kings.

Proof: Suppose that tournament T has two kings. We will show that in fact T has a third king. (Possibly even more.)

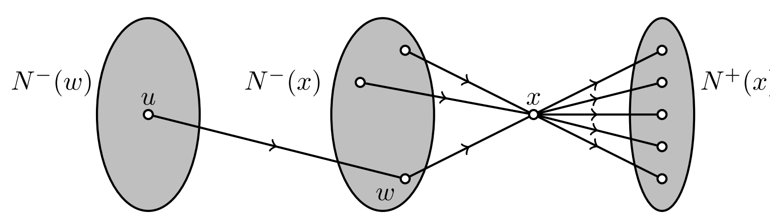
Call one of the kings x. Then x beats some other vertices, but since there is another king somewhere, someone beats x. Therefore $N^+(x) \neq \emptyset$ and $N^-(x) \neq \emptyset$.

First we claim that T has another king in $N^-(x)$: Among all the vertices in $N^-(x)$, choose $w \in N^-(x)$ such that $|N^+(w) \cap N^-(x)|$ is maximized. That is, w has the property that no vertex of $N^-(x)$ has more arrows to other vertices in $N^-(x)$ than does w. We claim that w is a king. Assume to the contrary that w is not a king. Notice that any vertex in $N^+(x)$ can be reached from w via a path of length 2, so there must be some vertex $y \in N^-(x)$ for which there is no w, y-path of length 1 or 2 (otherwise w would be a king).

Then then $w \not\to y$, so $y \to w$. (See the diagram, right.) Now, for any vertex $z \in N^+(w) \cap N^-(x)$, we must have $y \to z$, for otherwise there is a length-2 path $w \to z \to y$. This means that whenever $w \to z$, we also have $y \to z$. So y has at least as many arrows to other vertices of $N^-(x)$ as w does. But also $y \to w$, so in fact y has *more* arrows to other vertices of $N^-(x)$ than does w. This contradicts our choice of w. Hence w is a king.

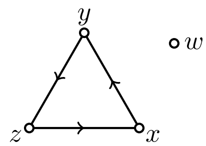


The previous paragraph says that whenever T has more than two kings, and x is a king, then there is a king w in $N^-(x)$. Applying this fact to w, we see that since w is a king, there is another king u in $N^-(w)$. (See drawing, right.) Therefore T has three kings x, w and u. ■

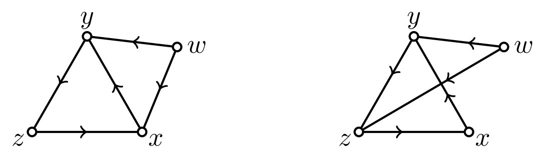


3. Let $n \in \mathbb{N}$. Prove that there is an n-vertex tournament in which every vertex is a king if and only if $n \notin \{2,4\}$.

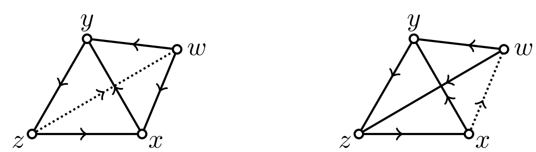
Proof: First let's show that if $n \in \{2,4\}$, then there is no n-vertex tournament in which every vertex is a king. Certainly if $n=2$, then the only n-vertex tournament is an orientation of K_2, and one vertex is a king and the other is not. Next, suppose there is a tournament T on $n=4$ vertices, for which all four vertices are kings. (We will reach a contradiction, proving no such tournament exists.) Take an arbitrary arc xy of T. Since y is a king it can reach x in two steps, so T has a vertex z and arcs yz and zx, as shown below. These three vertices induce a directed cycle in T.



Now consider the fourth vertex w. It is a king, so it must have positive out-degree. Without loss of generality, say the arc wy is present. Since w is a king, it must be one or two steps from x. Thus either T has an arc wz (shown below, left) or wx (shown below right).

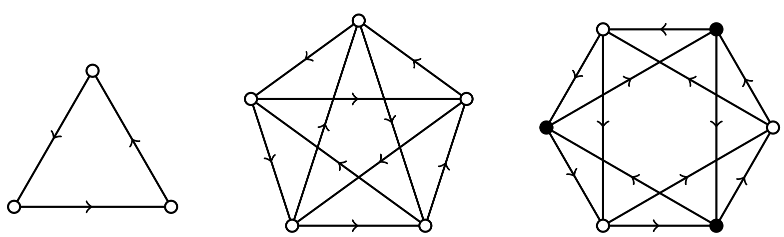


Now, w must be reachable from the other vertices (because they are kings) so the missing arc of the two above configurations must be as shown below (dotted). But now we have a contradiction, because x is not a king in the first case, and y is not a king in the second case.

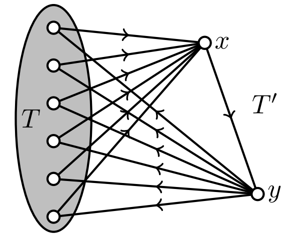


We conclude that there are no tournaments on 2 or 4 vertices for which all vertices are kings.

Next we will show that if $n \notin \{2,4\}$, then there is an n-vertex tournament for which all vertices are kings. First, this is trivially true for $n=1$, and it is true for $n=3$, $n=5$, and $n=6$, as shown below. For $n=6$ (right), note that any vertex has a path of length 1 or 2 to any other. So if the three missing arcs are filled in arbitrarily, then every vertex is a king in the resulting tournament.



To complete the proof, we will show that if there is an n-vertex tournament T in which all vertices are kings, then there is an $(n+2)$-vertex tournament T' in which all vertices are kings. Indeed, to construct T' from T, just add to T two new vertices x and y and an arc xy. Join arcs from all vertices of T to x, and join arcs from y to all vertices of T, as shown below.



Now any vertex in T' that happens to be a vertex of T is a king in T' because it reaches any vertex of T in two steps, plus it reaches x or y in one or two steps. Also, y is a king because it can reach x in two steps, and every other vertex in one step. Finally x is a king because it reaches y in one step and every other vertex in two steps. Thus every vertex of T' is a king.

Summary: we have shown that there are tournaments on 1, 3, 5 and 6 vertices in which every vertex is a king, and we have shown that if there is a n-vertex tournament in which every vertex is a king, then there is an $(n+2)$-vertex tournament in which every vertex is a king. This implies that there are n-vertex tournaments in which every vertex is a king for $n \in \{1,3,5,6,7,8,9,10,....\} = \mathbb{N} - \{2,4\}$. ■

5 Prove that every simple planar graph with at least four vertices has at least four vertices of degree less than 6.

Proof: Suppose G is a planar graph with at least four vertices. Take a planar embedding of it on the plane. If we add edges to G (with none crossing) until G becomes a maximal planar graph G', then we certainly have not decreased any vertex degrees. If we can prove that the maximal planar graph G' has at least four vertices of degree less than 6, then the same is true for G.

Therefore, for the rest of the proof it suffices to assume that G is maximal planar. Fix some maximal planar embedding of G. Say G has e edges and v vertices. Then because G is maximal planar we know

$$e = 3v - 6.$$

Now, for each non-negative integer i, let a_i be the number of vertices in G of degree equal to i. Because G is maximal planar and contains more than three vertices, it follows that

$$a_0 = a_1 = a_2 = 0.$$

(Reason: G has no vertex of degree 0, because if it did have one we could draw an edge from that vertex to another vertex on G, violating the fact that G is maximal planar. For the same reason, G has no vertex of degree 1. Finally G has no vertex x of degree 2, because it it did, x would lie on two faces, which are necessarily triangular [because G is maximal planar]. Then G would simply be a triangle, and this violates our assumption that G has four or more vertices.)

Let $d = \Delta(G)$. Then $v = a_3 + a_4 + a_5 + \cdots + a_d$ (recall $a_0 = a_1 = a_3 = 0$). The sum of the vertex degrees of G is then

$$\begin{aligned} 3a_3 + 4a_4 + 5a_5 + \cdots + da_d &= 2e \\ &= 2(3v-6) \\ &= 6v - 12 \\ &= 6(a_3 + a_4 + a_5 + \cdots + a_d) - 12 \\ &= 6a_3 + 6a_4 + 6a_5 + \cdots + 6a_d - 12 \end{aligned}$$

From this it follows that

$$12 + a_7 + 2a_8 + 3a_9 + \cdots + (d-6)a_d = 3a_3 + 2a_4 + a_5.$$

This implies

$$12 \le 3a_3 + 2a_4 + a_5 \le 3(a_3 + a_4 + a_5).$$

Since $a_0 = a_1 = a_2 = 0$, we get

$$12 \le 3(a_0 + a_1 + a_2 a_3 + a_4 + a_5),$$

that is

$$4 \le a_0 + a_1 + a_2 + a_3 + a_4 + a_5.$$

This says that the total number of vertices of degree less than 6 is at least 4, which is what we needed to prove. ■

10. Suppose G has v vertices, e edges, and its girth is g. Prove that $\gamma(G) \ge \frac{e}{2}\left(1 - \frac{2}{g}\right) - \frac{v}{2} + 1$.

Proof: Suppose that G is 2-cell embedded in a surface of genus $\gamma(G)$, with f faces, which we denotes as $F_1, F_2, \ldots, F_f$. Notice that any face F_i has length $\ell(F_i) \ge g$. By familiar arguments,

$$2e = \sum_{i=1}^{f} \ell(F_i) \ge \sum_{i=1}^{f} g = fg,$$

so $f \le \frac{2e}{g}$. Inserting this into Euler's formula, we get

$$\begin{aligned} v - e + f &= 2 - 2\gamma(G) \\ v - e + \frac{2e}{g} &\ge 2 - 2\gamma(G) \\ v - e\left(1 - \frac{2}{g}\right) &\ge 2 - 2\gamma(G) \\ 2\gamma(G) &\ge e\left(1 - \frac{2}{g}\right) - v + 2 \\ \gamma(G) &\ge \frac{e}{2}\left(1 - \frac{2}{g}\right) - \frac{v}{2} + 1. \end{aligned}$$

■

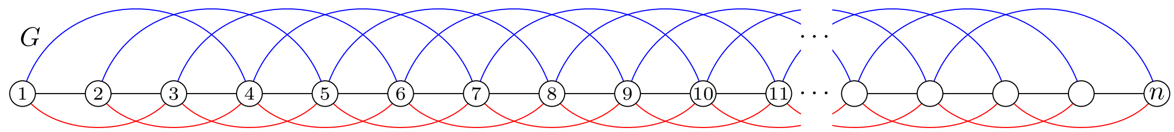
7 Let G be the graph with $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{v_i v_j : 1 \le |i-j| \le 3\}$. Prove that G is maximal planar.

Proof: For convenience, label the vertices $1, 2, 3, \dots, n$ instead of $v_1, v_2, \dots, v_n$. We can visualize G as a path P_n on n vertices with edges connecting vertices that are at distance 1, 2 or 3 from each other. Below is a picture. For clarity we color the edges joining vertices on P_n black if their endpoints are distance 1 apart on P_n, red if their vertices are at distance 2 apart on P_n and blue if their vertices are at distance 3 apart on P_n.

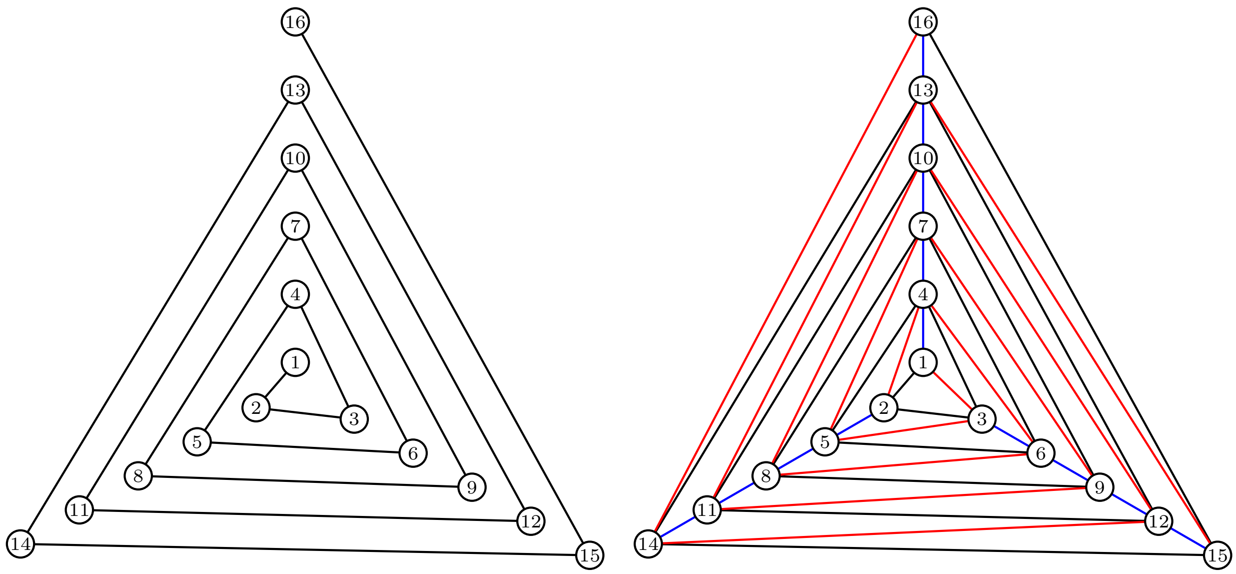
Notice that there are $n-1$ black edges, $n-2$ red edges, and $n-3$ blue edges. Therefore the total number of edges is

$$e = (n-1) + (n-2) + (n-3) = 3n - 6.$$

Because $e = 2n-6$, we know that if G is planar, then it is maximal planar. (Recall that if G is planar then $e \le 2n-6$, with equality holding if G is maximal planar.)

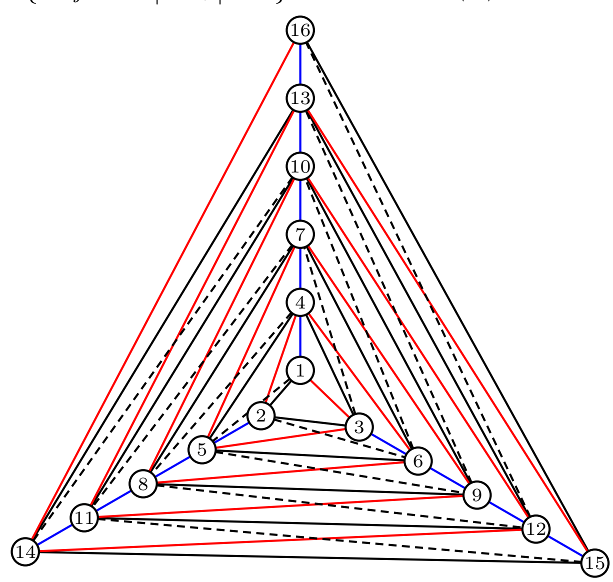


So we just need to show that G is planar. The trick is to draw the path P_n on a spiral, so its vertices are on radial lines spaced 120° apart, as illustrated below on the left. Then vertices on the same radial line are congruent modulo 3, so the blue lines connect them (see drawing below, right). The red lines are as indicated. Clearly this is a planar drawing, and all faces are triangles, so G is maximal planar. ■



8 Let G be the graph with $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{v_i v_j : 1 \le |i-j| \le 4\}$. Prove that $\nu(G) = n-4$.

Proof: Let G' be the graph from the previous problem, and note that G' is a subgraph of the current graph G. Moreover, G' had $3n-6$ edges, and the present G has an additional $n-4$ edges (which are at distance 4 on P_n). Therefore G has $(3n-6)+(n-4) = 4n-10$ edges, and G' is a maximal planar subgraph of G. By Proposition 6.3.13, $\nu(G) \ge e(G) - e(G') = (4n-10) - (3n-6) = n-4$. To confirm that $\nu(G) \le n-4$ here is a drawing of it with $n-4$ crossings. Note that the new dashed lines connecting vertices on P_n at distance 4 cross all but the first and last red edge. As there are $n-2$ red edges, there are $(n-2)-2 = n-4$ crossings. We have established $\nu(G) \le n-4$ and $\nu(G) \ge n-4$, so $\nu(G) = n-4$. ■



6 Prove that if G is planar and every face in a plane embedding of G has even length, then G is bipartite.

Proof: We will use induction on the number of faces of a plane embedding of G. For the basis case, assume G has just one face. Then G has no cycles (whether or not the face has even length), and a graph with no cycles is a forest, which is bipartite.

Let $m > 2$. Assume that if G' is any plane graph with fewer than m faces, and all the faces have even length, then G' is bipartite.

Now let G be a planar graph with m faces, and suppose G has a planar representation in which every face has even length. We need to show that G is bipartite.

Embed G on the plane so that all of the faces $F_1, F_2, F_3, \ldots, F_f$ have even length. Since there are more than two faces, G must have an an edge xy that separates one face from some other face. Without loss of generality, say xy separates F_1 from F_2. Form the graph $G' = G - xy$, that is, $V(G') = V(G)$ and $E(G') = E(G) - \{xy\}$. Then G' has faces $F_2', F_3, \ldots, F_f$, where $F_2' = F_1 \cup F_2$, but otherwise G and G' have the same faces. Now, G' has fewer than m faces. Further, we claim that every face of G' has even length. First, all the faces $F_3, \ldots, F_f$ have even length because they have even length in G. Also note that $\ell(F_2') = \ell(F_1) + \ell(F_2) - 2$, and because F_1 and F_2 have even length by assumption, so does F_2'. So every face of G' has even length, so G' is bipartite by the inductive hypothesis.

Note that the edge xy is a bridge of G, for otherwise it would be on just one face of G. Then the bipartite graph G' has an x, y-walk $xx_1x_2x_3 \ldots x_ny$ that avoids the edge xy and follows the boundary of F_1 in G, minus the edge xy. The length of this walk is one less than the length of F_1, so the length is odd. Therefore x and y are in different partite sets of G', so in adding the edge xy back to get G we arrive at a bipartite graph. ∎