Test One | Advanced Graph Theory | March 5, 2020
MATH 656
Name: ____________________ | R. Hammack | Score: ________

Directions: Choose any four questions. Each of your four chosen questions is 25 points, for a total of 100 points. If you do more than four questions, please clearly indicate which of the four you want to contribute toward your 100 points.

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1. Prove: A graph G is m-colorable if and only if $\alpha(G\Box K_m) \geq n(G)$. ($\alpha$ is the independence number.)

2. (a) Prove that there is no simple graph with 6 vertices and 13 edges that has chromatic number 3.
   (b) Give an example of a simple graph with 6 vertices and 12 edges that has chromatic number 3.

3. Given finite sets $S_1, S_2, \ldots, S_m$, form the set $U = S_1 \times S_2 \times \cdots \times S_m$. Let G be the graph with $V(G) = U$ and

$$E(G) = \big\{(x_1, x_2, \ldots x_m)(y_1, y_2, \ldots, y_m) \mid x_i \neq y_i \text{ for each } 1 \leq i \leq m\big\}.$$

   That is, two vertices are adjacent if and only if they differ in every coordinate. Determine $\chi(G)$.

4. Let G be a simple graph with n vertices. Recall that $\chi(G;k) = \sum_{r=1}^{n} p_r(G)k(k-1)(k-2)\cdots(k-r+1)$, where $p_r(G)$ is the number of partitions of $V(G)$ into r non-empty independent sets. Use this to prove that the coefficient of k^{n-1} in $\chi(G;k)$ is $-e(G)$.

5. Prove: If a graph G has c components, then $\chi(G;k) = k^c f(k)$, where $f(0) \neq 0$.

6. Prove: If G is a graph for which $\chi(G;k) = k^c f(k)$, where $f(0) \neq 0$, then G has c components.

7. (a) Prove that every n-vertex plane graph that is isomorphic to its dual has $2n-2$ edges.
   (b) For each $n \geq 4$, construct an n vertex plane graph that is isomorphic to its dual.

8. The 4-color theorem asserts that if G is planar, then $\chi(G) \leq 4$. Use this to prove that every planar graph decomposes into the union of two bipartite graphs. That is, prove that if G is planar, then there exist bipartite graphs A and B with $V(A) = V(B) = V(G)$, and $E(G) = E(A) \cup E(B)$, while $E(A) \cap E(B) = \emptyset$.

9. Prove that every simple planar graph with at least four vertices has at least four vertices of degree less than 6.

10. Prove that $3\nu(K_{n,n}) \leq \nu(K_{n,n,n}) \leq 3\binom{n}{2}^2$. ($\nu$ denotes crossing number.)