Test Two | Advanced Graph Theory | April 18, 2019

MATH 656

Name: ______________________ | R. Hammack | Score: ________

Directions: Choose any four questions. Each of your four chosen questions is 25 points, for a total of 100 points.
If you do more than four questions, please clearly indicate which of the four you want to contribute toward your 100 points.

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1. Prove that if G is a simple graph and $|E(G)| > \alpha'(G)\Delta(G)$, then G is Class 2.
(Recall that $\alpha(G')$ is the *edge independence number*, that is, the size of a maximum matching in G.)

2. Let G be a simple graph without isolated vertices. Prove that if the line graph $L(G)$ is connected and regular, then either G is regular, or G is a bipartite graph in which vertices in the same partite set have the same degree.

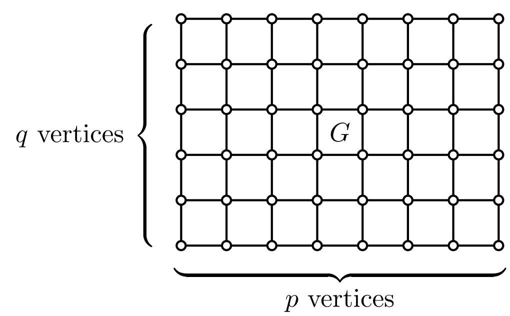
3. Let D be a digraph (loops allowed) such that $d^+_D(v) \leq d$ and $d^-_D(v) \leq d$ for all $v \in V(G)$. Prove that $E(D)$ can be colored using at most d colors, so that the edges entering each vertex have distinct colors and the edges exiting each vertex have distinct colors.

4. Prove that if G is Eulerian, then $L(G)$ is Eulerian.

5. Prove that every maximal plane graph other than K_4 is 3-face-colorable.

6. Prove that if G is a plane triangularization, then the planar dual G^* has a 2-factor (i.e. a 2-regular spanning subgraph).
(You may assume the Four Color Theorem.)

7. Given positive integers p, q, let G be the grid with p vertices on the horizontal side and q vertices on the vertical side.
Prove that G is Hamiltonian if and only if at least one of p and q is even.



8. Suppose that $M = (E, \mathbb{I})$ is a matroid. Prove that if $r(X) = r(X \cap Y)$, then $r(X \cup Y) = r(Y)$.
Does the converse necessarily hold?
(You are allowed to use the sub-modularity property of matroids: $r(X \cap Y) + r(X \cup Y) \leq r(X) + r(Y)$ for all $X, Y \subseteq E$.)

9. Prove that in a matroid $M = (E, \mathbb{I})$, a set $X \subseteq E$ is a hypobase if and only if it is a hyperplane.
(Recall: a hypobase is a maximal subset containing no base. A hyperplane is a maximal proper subspace.)
You can use the fact that $\sigma(\sigma(X)) = \sigma(X)$ in a matroid.