Test Two | Advanced Graph Theory | April 18, 2019
MATH 656
Name: ______________ | R. Hammack | Score: ______

Directions: Choose any four questions. Each of your four chosen questions is 25 points, for a total of 100 points. If you do more than four questions, please clearly indicate which of the four you want to contribute toward your 100 points.

1. Prove that if G is a simple graph and $|E(G)| > \alpha'(G)\Delta(G)$, then G is Class 2.
(Recall that $\alpha(G')$ is the *edge independence number*, that is, the size of a maximum matching in G.)

Proof: (Contrapositive) Suppose that G is not Class 2. Thus G is Class 1, meaning $\chi'(G) = \Delta(G)$. Select a proper edge coloring of G with $\Delta(G)$ colors. Then each vertex of G is incident with exactly one edge of each color, so each color class is a perfect matching. Thus $\alpha'(G)$ equals the number of edges in each of the $\chi'(G)$ color classes. Consequently $|E(G)| = \alpha'(G)\chi'(G) = \alpha'(G)\Delta(G)$. Thus $|E(G)| > \alpha'(G)\Delta(G)$ is untrue. ■

2. Let G be a simple graph without isolated vertices. Prove that if the line graph $L(G)$ is connected and regular, then either G is regular, or G is a bipartite graph in which vertices in the same partite set have the same degree.

Proof: Let $L(G)$ be regular of degree Δ, and connected. Consider a typical path $P = e_1, e_2, e_3, \ldots, e_n$ in $L(G)$, where vertices e_i and e_{i+1} are adjacent in $L(G)$ for $1 \leq i < n$. As $L(G)$ is regular, we have $d_{L(G)}(e_i) = \Delta$ for each e_i. In G, the e_i are edges, where e_i is incident to e_{i+1} for $1 \leq i < n$. We can write $e_1 = x_0x_1$, $e_2 = x_1x_2$, $e_3 = x_2x_3$, $\ldots$, $e_n = x_{n-1}x_n$. Each e_i is incident with $d_{L(G)}(e_i) = \Delta$ other edges of G, which means

$$d_G(x_{i-1}) + d_G(x_i) = \Delta + 2 \qquad \text{for each edge } e_i = x_{i-1}x_i.$$

Say $d_G(x_0) = d$. Using this and the above equation,

- $d_G(x_0) = d$,
- $d_G(x_1) = \Delta + 2 - d$
- $d_G(x_2) = d$
- $d_G(x_3) = \Delta + 2 - d$
- $d_G(x_4) = d$
- $d_G(x_5) = \Delta + 2 - d$
- $d_G(x_6) = d$, etc.

Because $L(G)$ is connected, we can reach any vertex $e_n \in V(L(G))$ by such a path P. Hence we can reach any vertex x_n of G as an endpoint of the final edge e_n.

Case 1: Suppose $d = (\Delta + 2)/2$ (so that $d = \Delta + 2 - d$). Then every vertex of G that we can reach from x_0 has degree d, which is to say that G is regular.

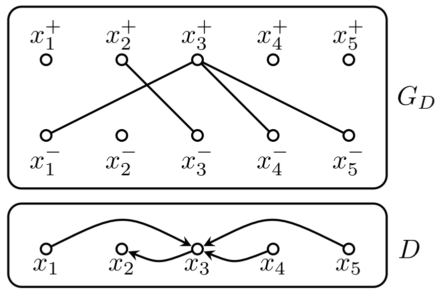
Case 2: Suppose $d \neq (\Delta + 2)/2$ (so that $d \neq \Delta + 2 - d$). Then any vertex of G either has degree d or degree $\Delta + 2 - d$. Let X be the set of vertices of degree d and let Y be the set of vertices of degree $\Delta + 2 - d$. Then $V(G) = X \cup Y$ and $X \cap Y = \emptyset$. An arbitrary edge $e = xy$ of G cannot have both endpoints in X because that would mean $d_{L(G)}(e) = 2d - 2 = \Delta$, violating $d \neq (\Delta + 2)/2$. Nor can e have both endpoints in Y, for that would mean $d_{L(G)}(e) = 2(\Delta - d + 2) - 2 = \Delta$, again violating violating $d \neq (\Delta + 2)/2$. Thus every edge of G joins X to Y, and hence G is bipartite, with partite sets X and Y. Moreover all vertices of one partite set have the same degree, and all vertices of the other partite set have the same degree, as desired. ■

3. Let D be a digraph (loops allowed) such that $d^+_D(v) \leq d$ and $d^-_D(v) \leq d$ for all $v \in V(G)$. Prove that $E(D)$ can be colored using at most d colors, so that the edges entering each vertex have distinct colors and the edges exiting each vertex have distinct colors.

Proof: Let D be as stated. From D we are going to make a graph G_D with twice as many vertices as D. For each vertex x of D, let G have two vertices x^- and x^+. That is, $V(G_D) = \{x^-, x^+ \mid x \in V(G)\}$. Also put

$$E(G_D) = \{x^- y^+ \mid xy \in E(D)\}.$$

In words, for any arc xy from x to y in the digraph D, there is an edge joining x^- to y^+ in G_D. Note that G_D is uniquely determined by D, and conversely D can be unambiguously reconstructed from G_D.

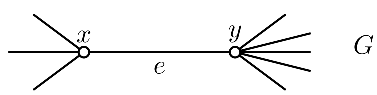


Notice that G_D is a bipartite graph with partite sets $\{x^- \mid x \in V(G)\}$ and $\{x^+ \mid x \in V(G)\}$. Moreover $d^+_D(x) = d_{G_D}(x^+)$ and $d^-_D(x) = d_{G_D}(x^-)$. Thus G_D is a bipartite graph with $\Delta(G) \leq d$. König's theorem (Theorem 7.1.7 in West) applies and gives $\chi'(G_D) = \Delta(G) \leq d$. Give G_D a proper edge coloring with at most d colors. Now color the arcs of D by giving each arc $xy \in E(D)$ the exact same color of x^-y^+ in G.

Then for any $x \in V(D)$ the arcs xy exiting x get the same colors as the edges x^-y^+ of G_D that are incident with x^-. Consequently the edges exiting x get distinct colors, and no more than d colors are used. Also for any $x \in V(D)$ the arcs yx entering x get the same colors as the edges y^-x^+ of G_D that are incident with x^+. Consequently the edges entering x get distinct colors, and no more than d colors are used. ■

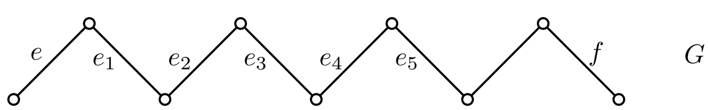
4. Prove that if G is Eulerian, then $L(G)$ is Eulerian.

Proof: Suppose that G is Eulerian, so it is connected and every vertex has even degree. We need to show that the same is true for $L(G)$. To begin, consider an arbitrary edge $e = xy$ of G, that is, and arbitrary vertex of $L(G)$.



Since x and y have even degree, e is incident with $d(x) - 1$ vertices at x, and $d(y) - 1$ vertices at y. Therefore e is incident with $d(x) - 1 + d(y) - 1 = d(x) + d(y) - 2$ other edges of G, and this is the degree of the vertex e in $L(G)$. Because $d(x)$ and $d(y)$ are even, $d(x) + d(y) - 2$ is even, and thus any vertex e of $L(G)$ has even degree.

Now we must show that $L(G)$ is connected. For any two edges $e, f \in E(G)$ choose a path in G whose edges are $e = e_0, e_1, e_2, \ldots, e_n = f$ that begins with e and ends with f (possible because G is connected). Then by definition of $L(G)$, the sequence of vertices $e = e_0, e_1, e_2, \ldots, e_n = f$ is a path joining e to f. Thus $L(G)$ is connected.



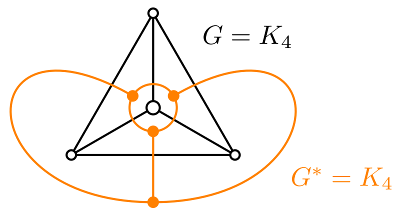
Because $L(G)$ is connected and all its vertices have even degree, it is Eulerian. ■

5. Prove that every maximal plane graph other than K_4 is 3-face-colorable.

   Proof: First note that K_4 cannot be 3-face-colored because each of its faces shares boundaries with all of the remaining four faces. Therefore whichever color we give a particular face, the three faces that share a boundary with it must be given three new and distinct colors.

   Suppose G is a maximal plane graph that is not K_4. In particular G is a plane triangularization. Regard the dual graph G^* as the planar graph formed by putting a vertex inside each triangular face of G and connecting two vertices x, y of G^* across across an edge e of G whenever e is the common boundary of the two faces of G that x and y are in.

   Notice that $K_3^* = K_3$, and in fact $G^* = K_3$ if and only if $G = K_3$, because $(G^*)^* = G$.

   Further note that in any event, G^* is 3-regular because every face of G is a triangle. Moreover, if we properly color the vertices of G^* with $\chi(G^*)$ colors, then that coloring induces a proper face-coloring of G if we give each face the color of the vertex of G^* that it contains. Therefore G can be properly $\chi(G^*)$-face-colored.

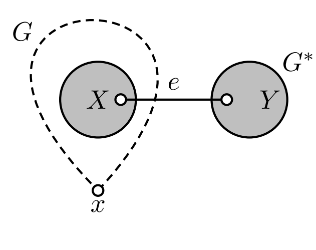
   Now, G^* is 3-regular, so it is certainly not an odd cycle. And if it were a complete graph then it could only be K_4, and we have expressly assumed that this is not the case. Thus Brook's theorem applies to G^*. (Recall that Brook's theorem states that $\chi(H) \leq \Delta(H)$ unless H is an odd cycle or a complete graph.) Applying Brook's theorem, we get $\chi(G^*) \leq \Delta(G^*) = 3$. Therefore G^* can be properly 3-vertex-colored, so G can be properly 3-face-colored. ■

6. Prove that if G is a plane triangularization, then the planar dual G^* has a 2-factor (i.e. a 2-regular spanning subgraph). (You may assume the Four Color Theorem.)

   Proof: Suppose G is a plane triangularization. By the Four Color Theorem, G has a proper vertex coloring with four colors. Form the dual G^* by putting a vertex in each face of G and connecting any pair of vertices that are in faces that share an edge. Then each face of G^* contains exactly one vertex of G. Color each face with the same color as the vertex contained in that face. This results in a 4-face-coloring of G^*. Thus G^* is 4-face colorable.

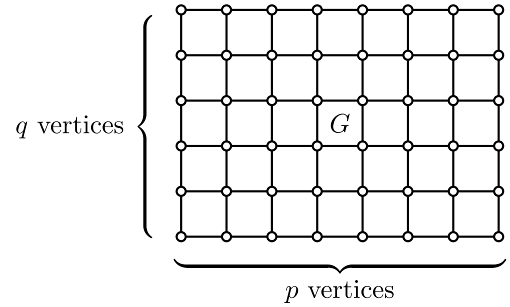
   Next we observe that because each face of G is a triangle, the dual G^* is 3-regular.

   Moreover, observe that G^* is 2-edge-connected. Indeed, if it had a bridge e, then e would have the same face on either side of it. Say that face contains the vertex x of G^*. Then $(G^*)^* = G$ has a loop at x, and this contradicts the fact that G is a plane triangularization.

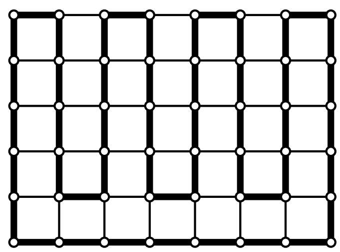
   

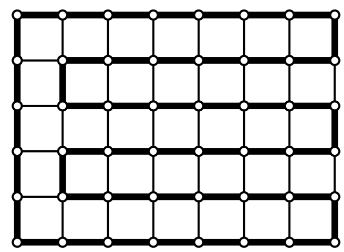
   In summary, G^* is a simple 3-regular 2-connected 4-face-colorable plane graph. By Tait's theorem, G^* is 3-edge colorable. Give G^* a proper edge coloring with 3 colors. Notice that each vertex of G^* is incident with edges of all three colors. Pick two of the colors, say red and green, and let H be the subgraph of G^* induced on the red and green edges. Then each vertex of H has degree 2, because it is incident with one red and one green edge. Thus H is 2-regular. Further H is spanning because every vertex of G^* is incident with a red and green edge. Consequently H is a 2-factor of G^*. ■

7. Given positive integers p, q, let G be the grid with p vertices on the horizontal side and q vertices on the vertical side. Prove that G is Hamiltonian if and only if at least one of p and q is even.



Proof: If p is even, then we can achieve a Hamiltonian cycle using the pattern on the left. If q is even, then we can achieve a Hamiltonian cycle using the pattern on the right.





Conversely suppose that it is not the case that at least one of p and q is even, that is, suppose that they are both odd. We need to prove that G has no Hamiltonian cycle. Suppose to the contrary that G does have a Hamiltonian cycle H. As usual, let f_i' be the number of length-i faces inside H, and let f_i'' be the number of length-i faces outside H. Every bounded face in the plane embedding has length 4, and the unbounded face has length $2(p-1)+2(q-1) = 2p+2q-4$. Grinberg's theorem says

$$\sum_{i=3}^{pq} (i-2)(f_i' - f_i'') = 0,$$

and this reduces to

$$\begin{aligned}
(4-2)(f_4' - f_4'') + (2p+2q-4-2)(f_{2p+2q-4}' - f_{2p+2q-4}') &= 0 \\
2(f_4' - f_4'') + (2p+2q-6)(0-1) &= 0 \\
2(f_4' - f_4'') &= 2p+2q-6 \\
f_4' - f_4'' &= p+q-3. \qquad (1)
\end{aligned}$$

Now, $f_4' + f_4''$ equals the total number of square faces of G, which is length $\times$ width $= (p-1)(q-1)$. Adding $f_4' + f_4'' = (p-1)(q-1)$ to both sides of (**??**) yields

$$\begin{aligned}
2f_4' &= (p-1)(q-1) + p + q - 3 \\
&= pq - p - q + 1 + p + q - 3 \\
&= pq - 2
\end{aligned}$$

Because p and q are both odd, the expression $pq-2$ is odd, which contradicts $2f_4'' = pq-2$. From this contradiction we conclude that G is not Hamiltonian.

8. Suppose that $M = (E, \mathbb{I})$ is a matroid. Prove that if $r(X) = r(X \cap Y)$, then $r(X \cup Y) = r(Y)$. Does the converse necessarily hold?

   (You are allowed to use the sub-modularity property of matroids: $r(X \cap Y) + r(X \cup Y) \leq r(X) + r(Y)$ for all $X, Y \subseteq E$.)

   Proof: Suppose $r(X) = r(X \cap Y)$. Then this combined with the sub-modularity property yields $r(X \cup Y) \leq r(Y)$. But because $Y \subseteq X \cup Y$, we have $r(Y) \leq r(X \cup Y)$. As $r(X \cup Y) \leq r(Y)$ and $r(Y) \leq r(X \cup Y)$, it follows that $r(X \cup Y) = r(Y)$.

   The converse does not necessarily hold. Let X and Y be two distinct bases of M. Then $r(X \cup Y) = r(E) = r(Y)$. However $X \cap Y$ consists of fewer than $|X|$ independent elements, so $r(X) > r(X \cap Y)$, that is, $r(X) \neq r(X \cap Y)$

9. Prove that in a matroid $M = (E, \mathbb{I})$, a set $X \subseteq E$ is a hypobase if and only if it is a hyperplane. (Recall: a hypobase is a maximal subset containing no base. A hyperplane is a maximal proper subspace.) You can use the fact that $\sigma(\sigma(X)) = \sigma(X)$ in a matroid.

   Proof: Suppose $X \subseteq E$ is a hypobase of M, so X is a maximal subset containing no base. Since X contains no base, we know that $\sigma(X) \neq E$. From this $\sigma(\sigma(X)) = \sigma(X) \neq E$, so $\sigma(X)$ contains no base (otherwise we would have $\sigma(\sigma(X)) = E$. Because X is a maximal set containing no base and X is contained in the set $\sigma(X)$ containing no base, it follows that $X = \sigma(X)$. This means that X is a subspace. In particular X is a maximal subspace containing no base. Thus X is a maximal proper subspace, in other words X is a hyperplane.

   Conversely, suppose X is a hyperplane, that is, X is a maximal proper subspace. (In particular $\sigma(X) = X$ because X is a subspace.) Then of course X is a proper subset of E and it contains no base because otherwise we would have $X = \sigma(X) = E$. We just need to show that X is a maximal subset of E containing no base. Thus suppose $X \subseteq Y \subseteq E$ and Y contains no base. (We need to show $X = Y$.) Then $\sigma(X) \subseteq \sigma(Y) \subseteq \sigma(E)$ which reduces to $X \subseteq \sigma(Y) \subseteq E$. But $\sigma(Y) \neq E$ because Y contains no base. Consequently $\sigma(\sigma(Y)) \neq E$, which means $\sigma(Y)$ contains no base. Now from $X \subseteq \sigma(Y)$ and the fact that X is a maximal set containing no base, we get $X = \sigma(Y)$. Now we have $X \subseteq Y \subseteq \sigma(Y) = X$, so $X = Y$. We have now shown that X is a maximal subset of E containing no base so, X is a hypobase. ∎