Name: ______________________________ **Quiz 20** MATH 300

November 21, 2024

Directions. This is a take-home quiz. You may consult the textbook, but do not get help from any other resource or person. Turn in the completed quiz at the beginning of class on Tuesday December 3.

1. Consider the functions $f, g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $f(m, n) = (3m - 4n, 2m + n)$ and $g(m, n) = (5m + n, m)$. Find the formulas for $g \circ f$ and $f \circ g$.

$$g \circ f(m,n) = g(f(m,n)) = g(3m-4n, 2m+n) = \Big(5(3m-4n)+2m+n, 3m-4n\Big) = (17m-19n, 3m-4n)$$

Therefore $\boxed{g \circ f(m,n) = (17m-19n, 3m-4n)}$

$$f \circ g(m,n) = f(g(m,n)) = f(5m+n, m) = \Big(3(5m+n)-4m, 2(5m+n)+m\Big) = (11m+3n, 11m+2n)$$

Therefore $\boxed{f \circ g(m,n) = (11m+3n, 11m+2n)}$

2. The function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by the formula $f(m, n) = (5m + 4n, 4m + 3n)$ is bijective. Find its inverse.

Note that $f(m, n)$ can be defined by right-multiplying (m, n) by the matrix $\begin{pmatrix} 5 & 4 \\ 4 & 3 \end{pmatrix}$, as follows:

$$f(m,n) = (m,n)\begin{pmatrix} 5 & 4 \\ 4 & 3 \end{pmatrix} = (5m + 4n, 4m + 3n).$$

By standard linear algebra, the inverse of this matrix is $\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}$. Therefore f^{-1} is

$$f^{-1}(m,n) = (m,n)\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix} = (-3m + 4n, 4m - 5n).$$

3. Given a function $f : A \to B$ and a subset $Y \subseteq B$, is $f(f^{-1}(Y)) = Y$ always true? Prove or give a counterexample.

   This is **false**. Here is a counterexample.

   Let $A = \{1\}$ and $B = \{2, 3, 4\}$, and let $f : A \to B$ be the function $f = \{(1, 2)\}$.
   If $Y = \{2, 3\} \subseteq B$, then $f(f^{-1}(Y)) = \{2\} \neq Y$.

4. Given a function $f : A \to B$ and subsets $W, X \subseteq A$, prove $f(W \cap X) \subseteq f(W) \cap f(X)$.

   Proof. Suppose $y \in f(W \cap X)$.
   This means there is an element $x \in W \cap X$ for which $f(x) = y$.
   Because $x \in W \cap X$, then certainly $x \in W$, and hence $f(x) \in f(W)$. That is, $y \in f(W)$.
   Because $x \in W \cap X$, then certainly $x \in X$, and hence $f(x) \in f(X)$. That is, $y \in f(X)$.
   Now, since y belongs to both $f(W)$ and $f(X)$, we have $y \in f(W) \cap f(X)$.

   We've shown that $y \in f(W \cap X)$ implies $y \in f(W) \cap f(X)$, hence $f(W \cap X) \subseteq f(W) \cap f(X)$. ■