MATH 300 Test #1 October 1, 2024

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Directions: Problems 1–3 on Page 1 are short-answer. For all other problems you must show your work. This test is closed-book and closed-notes. No calculators or other electronic devices.

1. (12 points)

(a) Let $X = \{\ldots -2, 8, 18, 28, 38, 48, 58, \ldots\}$. Write ~~X in set-builder notation.~~

$\boxed{X = \{-2+10n : n \in \mathbb{Z}\}}$ or $\boxed{X = \{8+10n : n \in \mathbb{Z}\}}$ etc

(b) $\{5n : n \in \mathbb{Z},\ n^2 \leq 16\} = \boxed{\{-20, -15, -10, -5, 0, 5, 10, 15, 20\}}$

(c) $\displaystyle\bigcup_{n\in\mathbb{N}} \{x \in \mathbb{R} : |x| > 1/n\} = \boxed{\mathbb{R} - \{0\}}$ or $\boxed{\{x \in \mathbb{R} : x \neq 0\}}$

2. (12 points) Suppose A and B are sets for which $|A| = m$ and $|B| = n$. Find the cardinalities:

(a) $|\mathscr{P}(A) - \{A\}| = \boxed{2^m - 1}$

(b) $|\mathscr{P}(A) \times \mathscr{P}(A \times B)| = 2^m 2^{mn} = \boxed{2^{m+mn}} = \boxed{2^{m(1+n)}}$

(c) $\big|\{X \in \mathscr{P}(B) : |X| = 5\}\big| = \boxed{\binom{n}{5}}$

3. (4 points)

(a) Here are the first several rows of Pascal's triangle. Write the next row.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

(b) Use part (a) to find the coefficient of x^3y^3 in $(2x-y)^6$. Simplify your answer as much as possible.

$$20(2x)^3(-y^3) = -20 \cdot 8\, x^3 y^3 = -160\, x^3 y^3$$

Thus the coefficient is $\boxed{-160}$

4. (12 points) This question concerns the following statement.

For every real number x, there is a real number y for which $xy > x$.

(a) Is this statement true or false? Explain.

False because if $x = 0$, there is no y for which $xy > x$, that is, $0y > 0$.

(b) Write the statement in symbolic form.

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy > x$$

(c) Form the negation of your answer from (b) above, and simplify.

$$\sim(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy > x)$$

$$= \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \sim(xy > x)$$

$$= \boxed{\exists x \in \mathbb{R}, \forall y \in \mathbb{R}\ xy \leq x}$$

(d) Write the negation from (c) above as a well-formed English sentence.

There is a real number x with the property that $xy \leq x$ for every real number y.

5. (10 points) How many 10-digit integers have fewer than four 0's?

3 0's: $\leftarrow \binom{9}{3} 9^7$ of these

2 0's $\leftarrow \binom{9}{2} 9^8$ of these

1 0 $\leftarrow \binom{9}{1} 9^9$ of these

no 0's $\leftarrow 9^{10}$ of these

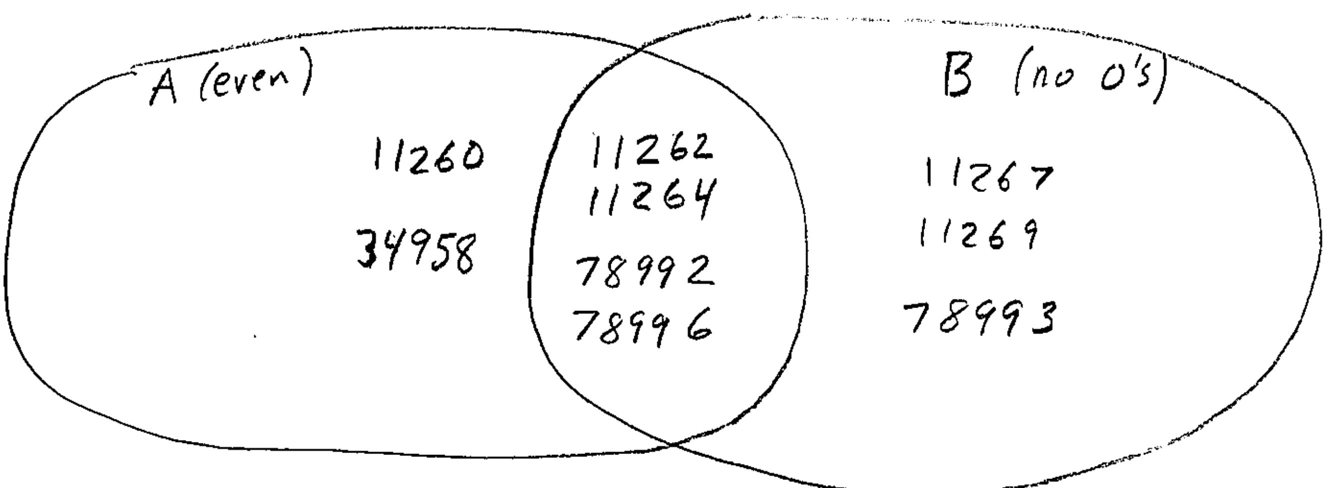
9 9 9 9 9 9 9 9 9 9

(Note: a 0 can't go in the leading position because otherwise we wouldn't have a 10-digit number!)

By the addition principle, the answer is

$$\boxed{\binom{9}{3}9^7 + \binom{9}{2}9^8 + \binom{9}{1}9^9 + 9^{10}}$$

6. (10 points) How many 5-digit positive integers are there that are even or contain no 0's?



$|A| = 9 \cdot 10 \cdot 10 \cdot 10 \cdot 5$ $\quad |B| = 9\ 9\ 9\ 9\ 9$ $\quad |A \cap B| = 9 \cdot 9 \cdot 9 \cdot 9 \cdot 4$

Answer: $|A \cup B| = |A| + |B| - |A \cap B|$

$$= \boxed{45000 + 9^5 - 4 \cdot 9^4}$$

7. (10 points) Suppose $x \in \mathbb{Z}$. Prove: If $x^2 - 6x + 5$ is even, then x is odd. [Use contrapositive.]

Proof Suppose x is not odd.
Then x is even, so $x = 2a$ for some $a \in \mathbb{Z}$.
So $x^2 - 6x + 5 = (2a)^2 - 6(2a) + 5 = 4a^2 - 12a + 4 + 1$
$= 2(2a^2 - 6a + 2) + 1$.
We have shown that $x^2 - 6x + 5 = 2b + 1$ for $b = 2a^2 - 6a + 2$. Therefore $x^2 - 6x + 5$ is odd. ∎

8. (10 points) Prove that $\sqrt{2}$ is irrational. [Use contradiction.]

Proof Suppose for the sake of contradiction that $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{a}{b}$ for integers a and b. We may assume this fraction is fully reduced, so a and b are not both even. Observe that

$$\sqrt{2} = \frac{a}{b}$$

$$\sqrt{2}^2 = \left(\frac{a}{b}\right)^2$$

$$2b^2 = a^2. \quad (*)$$

Consequently a^2 is even, which means a is even, so b must be odd. But a even means $a = 2n$ for some $n \in \mathbb{Z}$. Putting this into equation $(*)$ gives $2b^2 = (2n)^2$, or $2b^2 = 4n^2$, so $b^2 = 2n^2$. Hence b^2 is even, so b is even.
Thus b is both even and odd, a contradiction ∎

9. (10 points) Prove: If a and b are integers, then $(a+b)^3 \equiv a^3 + b^3 \pmod 3$. [Use direct proof]

Proof Assume a, b are integers.
Then by the binomial theorem,
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
Then $(a+b)^3 - (a^3+b^3) = 3a^2b + 3ab^2 = 3(a^2b+ab^2)$.
From this, $3 \mid ((a+b)^3 - (a^3+b^3))$ which implies $(a+b)^3 \equiv a^3+b^3 \pmod 3$ ∎

10. (10 points) Prove: If $n \in \mathbb{Z}$, then $4 \mid n^2$ or $4 \mid (n^2+3)$.

Proof (Direct) Assume $n \in \mathbb{Z}$

Case I Suppose n is even. Then $n = 2a$ for some $a \in \mathbb{Z}$, and $n^2 = 4a^2$. This means $4 \mid n^2$.

Case II Suppose n is odd. Then $n = 2a+1$ for some $a \in \mathbb{Z}$ and $n^2 + 3 = (2a+1)^2 + 3$
$= 4a^2 + 4a + 1 + 3 = 4a^2 + 4a + 4 = 4(a^2+a+1)$.
This means $4 \mid (n^2+3)$.

The cases above show $4 \mid n^2$ or $4 \mid (n^2+3)$. ∎