Introduction to Mathematical Reason — MATH 300 Test 2 — October 31, 2024

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1. Prove that x is odd if and only if x^2 is odd.

Proof First we use direct proof to prove that if x is odd, then x^2 is odd. Assume x is odd. Then $x = 2a+1$ for some $a \in \mathbb{Z}$. Consequently $x^2 = (2a+1)^2 = 4a^2+4a+1 = 2(2a^2+2a)+1$. Therefore $x^2 = 2b+1$, where $b = 2a^2+2a \in \mathbb{Z}$, Consequently x^2 is odd.

Conversely we will show that if x^2 is odd, then x is odd. For this we use contrapositive proof, Suppose x is not odd. Then x is even, so $x = 2a$ for some $a \in \mathbb{Z}$. Then $x^2 = (2a)^2 = 4a^2 = 2(2a^2)$, so x^2 is even. Therefore x^2 is not odd ■

2. Prove: There is a set X for which $\mathbb{N} \in X$ and $\mathbb{N} \subseteq X$.

Proof Let $X = \{\mathbb{N}\} \cup \mathbb{N} = \{\mathbb{N}, 1, 2, 3, 4, \ldots\}$

$$= \{\{1,2,3,4,\ldots\}, 1, 2, 3, 4, \ldots\}$$

Then $\mathbb{N} \in X$ and $\mathbb{N} \subseteq X$.

3. Suppose $A = \{6a + 15b : a, b \in \mathbb{Z}\}$ and $B = \{x \in \mathbb{Z} : 3 \mid x\}$. Prove that $A = B$.

Proof First we will show that $A \subseteq B$.
Suppose $x \in A$. Then $x = 6a + 15b$ for some $a, b \in \mathbb{Z}$, which means $x = 3(2a + 5b)$, and hence $3 | a$ from which it follows that $x \in B$. Therefore $A \subseteq B$.

Next we show that $B \subseteq A$. Let $x \in B$. Then $3 | x$ so $x = 3b$ for some $c \in \mathbb{Z}$. Then $x = 3b =$
$= 3 \cdot b \cdot 1 = 3b(-4+5) = -12b + 15b = 6(-2c) + 15b$.
That is, $x = 6(-2c) + 15b = 6a + 15b$ (where $a = -2c$).
Because $x = 6a + 15b$ for $a, b \in \mathbb{Z}$, it follows that $x \in A$. This establishes that $B \subseteq A$.

Since $A \subseteq B$ and $B \subseteq A$, it follows that $A = B$. ▨

4. Suppose R is a symmetric and transitive relation on a set A, and there is an element $a \in A$ for which aRx for every $x \in A$. Prove that R is reflexive.

Proof (Direct) Suppose R is transitive and symmetric and there is an element $a \in A$ for which aRx for every $x \in A$.

We now show that R is reflexive, that is, xRx for all $x \in A$.

Because aRx for all $x \in A$ and R is symmetric, it follows that xRa for all $x \in A$. In other words, $(xRa) \wedge (aRx)$ for all x in A. But R is transitive, so for all $x \in A$ we have

$$(xRa) \wedge (aRx) \Longrightarrow xRx.$$

Therefore xRx for all $x \in A$, which means R is reflexive. ▨

5. Suppose $n \in \mathbb{N}$. Use induction to prove that $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

Proof (Induction)

If $n=1$, this statement is $1^3 = \frac{1^2(1+1)^2}{4}$, which simplifies to the (true) statement $1 = \frac{2^2}{4}$.

Now suppose the statement is true for some $n = k \geq 1$, that is, suppose $1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$.

Then: $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3$

$$= \frac{k^2(1+k)^2}{4} + (k+1)^3$$

$$= \frac{k^2(1+k)^2 + 4(k+1)^3}{4}$$

$$= \frac{(1+k)^2\left(k^2 + 4(k+1)\right)}{4}$$

$$= \frac{(1+k)^2(k^2+4k+4)}{4}$$

$$= \frac{(1+k)^2(k+2)^2}{4} = \frac{(k+1)^2(k+1+1)^2}{4}.$$

We have shown that

$$1^3 + 2^3 + 3^3 + \cdots k^3 + (k+1)^3 = \frac{(k+1)^2(k+1+1)^2}{4},$$

that is, the statement is true for $n = k+1$.

This completes the proof by induction $\blacksquare$

6. Use induction or smallest counterexample to prove that $24 \mid (5^{2n} - 1)$ for every integer $n \geq 0$.

Proof (Induction)

① If $n=0$ the statement is $24 \mid (5^{2\cdot 0} - 1)$ which reduces to $24 \mid 0$, which is true.

② Now assume the statement is true for some $n = k > 1$, that is, assume $24 \mid (5^{2k} - 1)$. This means $5^{2k} - 1 = 24c$ for some $c \in \mathbb{Z}$.

Then
$$5^2(5^{2k} - 1) = 5^2 \cdot 24c$$
$$5^{2k+2} - 25 = 25 \cdot 24c$$
$$5^{2k+2} - 25 + 24 = 25 \cdot 24c + 24$$
$$5^{2(k+1)} - 1 = 24(25c + 1)$$

Thus $24 \mid (5^{2(k+1)} - 1)$, so the statement is true for $n = k+1$.

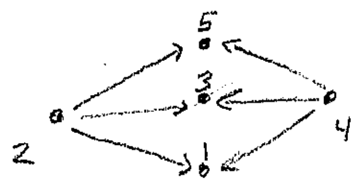
This completes the proof by induction ■

7. Prove that if X and Y are sets and $\mathscr{P}(X) \subseteq \mathscr{P}(Y)$, then $X \subseteq Y$.

Proof (Direct) Suppose X and Y are sets and $\mathscr{P}(X) \subseteq \mathscr{P}(Y)$. We will show that $X \subseteq Y$. Suppose $x \in X$. Then $\{x\} \subseteq X$, so $\{x\} \in \mathscr{P}(X)$ and because $\mathscr{P}(X) \subseteq \mathscr{P}(Y)$ it follows that $\{x\} \in \mathscr{P}(Y)$. This means $\{x\} \subseteq Y$, that is, $x \in Y$.

8. Consider the relation on the set $A = \{1, 2, 3, 4, 5\}$ defined as $R = \{(x, y) : x \text{ is even and } y \text{ is odd}\}$.

(a) Draw a diagram for R.



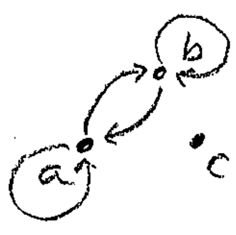
(b) Is R transitive? Explain.

No However we pick $x, y, z \in A$, the statement $(xRy) \wedge (yRz)$ is false so $(xRy) \wedge (yRz) \Rightarrow xRz$ is automatically TRUE.

9. Prove or disprove: Every transitive and symmetric relation is reflexive.

This is false. For a counterexample consider the following relation on $\{a, b, c\}$:

$R = \{(a,a), (a,b), (b,a) (b,b)\}$



This is transitive and symmetric, but not reflexive