Introduction to Mathematical Reason	MATH 300 Test #2	April 8, 2010
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PART I. Prove the following statements.

1. Prove that an integer a is even if and only if $a^2 + 2a + 9$ is odd.

Proof. First we will show that if a is even, then $a^2 + 2a + 9$ is odd. We use direct proof.
Suppose a is even. Then $a = 2k$ for some integer k, and

$$a^2 + 2a + 9 = (2k)^2 + 2(2k) + 9 = 4k^2 + 4k + 8 + 1 = 2(2k^2 + 2k + 4) + 1.$$

This shows that $a^2 + 2a + 9$ is twice an integer plus 1, so it is odd.

Conversely, we will show that if $a^2 + 2a + 9$ is odd, then a is even.
We use contrapositive proof; that is we will assume a is not even and show $a^2 + 2a + 9$ is not odd.
Suppose a is not even, so it is odd, and thus $a = 2k + 1$ for some integer k. Then

$$\begin{aligned} a^2 + 2a + 9 &= (2k + 1)^2 + 2(2k + 1) + 9 \\ &= 4k^2 + 4k + 1 + 4k + 2 + 9 \\ &= 4k^2 + 8k + 12 \\ &= 2(2k^2 + 4k + 6). \end{aligned}$$

This shows that $a^2 + 2a + 9$ is twice an integer, so it is even.

The proof is now complete. ■

2. Suppose A, B and C are nonempty sets. Prove that if $A \times B \subseteq B \times C$, then $A \subseteq C$.

Proof. We will use direct proof. Suppose $A \times B \subseteq B \times C$.

In what follows we show $A \subseteq C$.
Suppose $a \in A$.
Since B is not empty, there is an element $b \in B$, so $(a, b) \in A \times B$. (By definition of $\times$.)
But since $A \times B \subseteq B \times C$, it follows that $(a, b) \in B \times C$. (By definition of $\subseteq$.)
In particular, this gives us $a \in B$, so it now follows that $(a, a) \in A \times B$. (By definition of $\times$.)
But again, since $A \times B \subseteq B \times C$, it we get $(a, a) \in A \times C$. (By definition of $\subseteq$.)
In particular, this means $a \in C$. (By definition of $\times$.)

We've now shown $a \in A$ implies $a \in C$, so $A \subseteq C$. ■

3. Use induction to prove that $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

Proof: (Mathematical Induction)

(1) When $n = 1$ the statement is $1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$, which is true.

(2) Now assume the statement is true for some integer $n = k \geq 1$, that is assume $1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 = \dfrac{k^2(k+1)^2}{4}$.
Observe that this implies the statement is true for $n = k+1$, as follows:

$$\begin{aligned}
1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 &= \\
(1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3) + (k+1)^3 &= \\
\frac{k^2(k+1)^2}{4} + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1)^1)}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2((k+1)+1)^2}{4}
\end{aligned}$$

Therefore $1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 = \dfrac{(k+1)^2((k+1)+1)^2}{4}$, which means the statement is true for $n = k+1$.

This completes the proof by mathematical induction. ■

4. There exists a set X for which $\mathbb{Z} \in X$, $\mathbb{N} \in \mathscr{P}(X)$ and $\mathbb{R} \in \mathscr{P}(X)$.

Proof. Consider the set $X = \{\mathbb{Z}\} \cup \mathbb{R}$.
(That is, X contains every real number, *and* it also contains the *set* of all integers.)
We have $\mathbb{N} \subseteq X$ and $\mathbb{R} \subseteq X$, and this means $\mathbb{N} \in \mathscr{P}(X)$ and $\mathbb{R} \in \mathscr{P}(X)$.
Also, we have $\mathbb{Z} \in \{\mathbb{Z}\}$, so $\mathbb{Z} \in \{\mathbb{Z}\} \cup \mathbb{R} = X$. ■

5. Use induction to prove that $24|(5^{2n}-1)$ for every integer $n \geq 0$.

Proof. The proof is by mathematical induction.

(1) For $n = 0$, the statement is $24|(5^{2\cdot 0}-1)$. This simplifies to $24|0$, which is true.

(2) Now assume the statement is true for some integer $n = k \geq 1$, that is assume $24|(5^{2k}-1)$.
This means $5^{2k} - 1 = 24a$ for some integer a, and from this we get $5^{2k} = 24a + 1$.
Now observe that

$$\begin{aligned} 5^{2(k+1)} - 1 &= \\ 5^{2k+2} - 1 &= \\ 5^2 5^{2k} - 1 &= \\ 5^2(24a+1) - 1 &= \\ 25(24a+1) - 1 &= \\ 25 \cdot 24a + 25 - 1 &= 24(25a+1) \end{aligned}$$

This shows $5^{2(k+1)} - 1 = 24(25a+1)$, which means $24|5^{2(k+1)} - 1$.

This completes the proof by mathematical induction. ■

PART II. (10 points each) Decide if the following statements are true or false. Prove the true statements; disprove the false ones.

6. If A, B and C are sets, then $A \cup (B - C) = (A \cup B) - (A \cup C)$.

   This is FALSE. Here is a counterexample:

   Let $A = B = C = \{1\}$.
   Then $A \cup (B - C) = \{1\}$.
   Also $(A \cup B) - (A \cup C) = \emptyset$.
   This example shows that it is not always true that $A \cup (B - C) = (A \cup B) - (A \cup C)$.

7. Suppose a and b are integres. If $a|b$ and $b|a$, then $a = b$.

   This is FALSE. Here is a counterexample:

   Let $a = 2$ and $b = -2$.
   Then $a|b$ and $b|a$, but $a \neq b$.

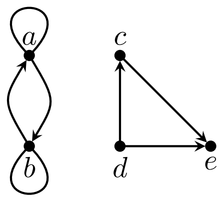
8. If A, B, C are sets and $A \cap B \cap C = \emptyset$, then $|A \cup B \cup C| = |A| + |B| + |C|$.

   This is FALSE. Here is a counterexample:
   Let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{3, 1\}$.
   Then $|A \cup B \cup C| = |\{1, 2, 3\}| = 3 \neq 6 = |A| + |B| + |C|$.

PART III. (10 points each)

9. Let $A = \{a, b, c, d, e\}$. Consider the relation $R = \{(a, a), (a, b), (b, a), (b, b), (d, c), (d, e), (c, e)\}$ on A.

   (a) Draw a diagram for the relation R.



   (b) Is the relation R reflexive? .... NO. For example, $(c, c) \notin R$.

   (c) Is the relation R symmetric? .... NO. For example, $(c, e) \in R$ but $(e, c) \notin R$.

   (d) Is the relation R transitive? .... YES. Whenever xRy and yRz, then also xRz.

10. Let n be a fixed positive integer. As noted in class, congruence modulo n is a relation on the set $\mathbb{Z}$. Prove that this relation is transitive.

Proof. We need to show that if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.
We will prove this conditional statement with direct proof.

Suppose that $a, b, c \in \mathbb{Z}$, and $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$.
This means $n \mid (a - b)$ and $n \mid (b - c)$.
Therefore $a - b = nk$ and $b - c = n\ell$ for integers k and ℓ.
Adding, we get $(a - b) + (b - c) = nk + n\ell$.
Simplifying, $a - c = n(k + \ell)$.
Consequently $n \mid (a - c)$.
Therefore $a \equiv c \pmod{n}$.

We have now shown that if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.
Consequently, the relation is transitive. ■