MATH 300 FINAL EXAM May 7, 2006

Name: ______________ R. Hammack Score: ______

**Directions:** Prove the following statements in the space provided. To get full credit you must show all of your work. Use of calculators is **not** allowed on this test.

1. Prove that if $A, B$ and $C$ are nonempty sets and $A \times B = A \times C$, then $B = C$.

2. Prove that if $x$ and $y$ are real numbers that are both greater than zero, then $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$. (Suggestion: consider proof by contradiction or contrapositive.)

3. Suppose $x \in \mathbb{Z}$. Prove $7x - 3$ is even if and only if $x$ is odd.

4. Prove or disprove: If $A$ and $B$ are nonempty sets, then $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$.

5. Prove or disprove: If $A$ and $B$ are sets with $A \neq B$, and $A \subseteq B$, then $|A| < |B|$.

6. Prove or disprove: If an equivalence relation on a set $A$ has finitely many equivalence classes, then $A$ is finite.

7. Suppose $a, b$ and $c$ are integers. Prove that if $a|b$ and $a|(b+c)$, then $a|c$.

8. Suppose $A$ and $B$ are sets. Use the technique of contrapositive proof to prove the following:
   If $A \times B = \emptyset$, then $A = \emptyset$ or $B = \emptyset$.

9. Prove that if $a \equiv 1 \pmod{5}$, then $a^2 \equiv 1 \pmod{5}$.

10. Prove that $\sqrt{2}$ is irrational.

11. Suppose $R$ is a transitive relation on a set $A$, and $x \not R\, x$ for all $x \in A$. Show that if $xRy$, then $y \not R\, x$.

The questions on this page involve the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $f((x, y)) = (x + y, x)$

12. Prove that $f$ is injective.

13. Prove that $f$ is surjective.

14. Find a formula for $f^{-1}$

15. Use mathematical induction to prove $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{N}$.

16. Use mathematical induction to prove $4|(5^n - 1)$ for every $n \in \mathbb{N}$.

17. Use mathematical induction to prove $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$ for every $n \in \mathbb{N}$.

18. Let $\mathbb{R}^+ = \{x \in \mathbb{R} \,:\, x > 0\}$. Prove that $|\mathbb{R}^+| = |\mathbb{R}|$.
(Suggestion: use the definition of what it means for two sets to have the same cardinality, combined with your knowledge of algebra and functions)

19. This problem concerns 4-letter codes that can be made from the letters A,B,C,D,E, ..., Z of the English Alphabet.

(a) How many such codes can be made?

(b) How many such codes are there that have no two consecutive letters the same?

20. How many 9-digit numbers can be made from the digits $1, 2, 3, 4, 5, 6, 7, 8, 9$ if repetition is not allowed and all the odd digits occur first (on the left) followed by all the even digits? (i.e. 1375980264 is such a number, but 0123456789 is not.)