European Journal of Combinatorics 47 (2015) 123–140

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

A prime factor theorem for bipartite graphs

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ARTICLE INFO

Article history:
Received 1 October 2013
Accepted 4 February 2015
Available online 2 March 2015

ABSTRACT

It has long been known that the class of connected nonbipartite graphs (with loops allowed) obeys unique prime factorization over the direct product of graphs. Moreover, it is known that prime factorization is not necessarily unique in the class of connected bipartite graphs.

But any prime factorization of a connected bipartite graph has exactly one bipartite factor. It has become folklore in some circles that this prime bipartite factor must be unique among all factorings, but until now this conjecture has withstood proof.

This paper presents a proof. We show that if a connected bipartite graph G has two factorings $G \cong A \times B$ and $G \cong A' \times B'$, where B and B' are prime and bipartite, then $B \cong B'$.

1. Introduction

We assume our reader is familiar with graph products, but we review the main definitions here to synchronize notation and terminology. See [4] for a survey.

Let Γ be the set of (isomorphism classes of) graphs without loops; thus $\Gamma \subset \Gamma_0$, where Γ_0 is the set of graphs with loops allowed. The *direct product* of two graphs $A, B \in \Gamma_0$ is the graph $A \times B$ with vertices $V(A) \times V(B)$ and edges

$$E(A \times B) = \left\{(a, b)(a', b') \mid aa' \in E(A) \text{ and } bb' \in E(B)\right\}.$$

Fig. 1 shows an example. This product is commutative and associative in the sense that the maps $(a, b) \mapsto (b, a)$ and $(a, (b, c)) \mapsto ((a, b), c)$ are isomorphisms $A \times B \to B \times A$ and $A \times (B \times C) \to (A \times B) \times C$, respectively.

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http://dx.doi.org/10.1016/j.ejc.2015.02.003
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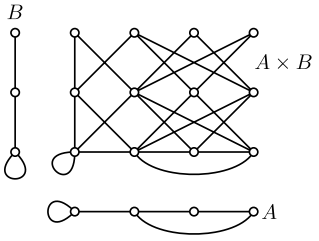


Fig. 1. Direct product of graphs.

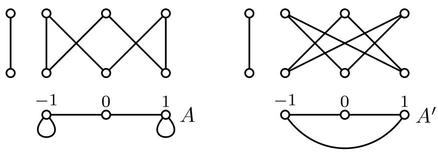


Fig. 2. Two prime factorings $A \times K_2$ and $A' \times K_2$ of the bipartite graph $G = C_6$.

If K_1^* denotes a vertex on which there is a loop, then $K_1^* \times A \cong A$ for any graph A, so K_1^* is the unit for the direct product. A nontrivial graph $G \in \Gamma_0$ is *prime over* $\times$ if for any factoring $G \cong A \times B$ into graphs $A, B \in \Gamma_0$ it follows that one of A or B is K_1^* and the other is isomorphic to G.

A consequence of a fundamental result by McKenzie [6] is that every connected nonbipartite graph in Γ_0 factors over $\times$ uniquely into primes in Γ_0. Specifically, if $G = A_1 \times A_2 \times \cdots \times A_k$ and $G = B_1 \times B_2 \times \cdots \times B_\ell$ are two prime factorings of a connected nonbipartite graph G, then $k = \ell$ and $B_i \cong A_{\pi(i)}$ for some permutation π of $\{1, 2, \ldots, k\}$. McKenzie's paper involves general relational structures; for purely graph-theoretical proofs of unique prime factorization, see Imrich [5], or [4] for a more recent proof.

But if G is bipartite, its prime factorization may not be unique. Fig. 2 illustrates this. It shows a graph $G = C_6$ and prime factorings $G \cong A \times K_2$ and $G \cong A' \times K_2$ with $A \ncong A'$.

Indeed, we claim that if B is any prime bipartite graph, there exists a bipartite graph G with distinct prime factorings, but with common bipartite prime factor B. To see this, say B has a bipartition $V(B) = X_0 \cup X_1$ and consider graphs $G = A \times B$ and $A' \times B$, where A and A' are the (prime) graphs from Fig. 2. To establish the claim we assert that there is an isomorphism $\varphi : A \times B \to A' \times B$. Indeed,

$$\varphi(a, b) = \begin{cases} (a, b) & \text{if } b \in X_0 \\ (-a, b) & \text{if } b \in X_1 \end{cases}$$

is such an isomorphism, as the reader is invited to check.

Let us examine factorings of bipartite graphs in more detail. An oft-used theorem by Weichsel states the following: Let A and B be connected graphs. Then $A \times B$ is connected if and only if at least one of A or B is not bipartite; if both A and B are bipartite, then $A \times B$ has exactly two components. Moreover, $A \times B$ is bipartite if and only if at least one factor is bipartite. (See Theorem 5.9 of [4] and its proof.)

It follows that if a connected bipartite graph G has a prime factoring $G = A_1 \times A_2 \times \cdots \times A_k$, then *exactly one* prime factor is bipartite. This is borne out in Fig. 2; but notice that although the prime factorings of $G = C_6$ are different, the bipartite factor $B = K_2$ is the same. Examples such as this one have prompted the following conjecture.

Conjecture 1. *Given two prime factorings of a connected bipartite graph, the prime bipartite factors are isomorphic.*

The origins of this conjecture are unclear, but it has circulated in the product graph community for some time. The article [2] proves it in the special case of graphs that have a bipartite factor of K_2. We here offer a completely general proof. We will show that if a connected bipartite graph factors as

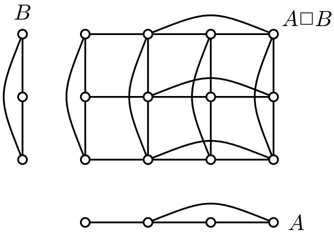


Fig. 3. Cartesian product of graphs.

$G \cong A \times B$ and $G \cong A' \times B'$, where B and B' are prime and bipartite, then $B \cong B'$. Doing this involves some machinery, which the next three sections review. Section 2 reviews the Cartesian product of graphs, and theorems regarding unique factorization over this product. Section 3 discusses the notion of so-called R-thinness, an important ingredient of our proof. Section 4 lays out the elements of the Cartesian skeleton operator on graphs, which converts certain questions about the direct product to questions about the (more easily understood) Cartesian product. All of these things are used to prove the main result in Section 5.

Parts of this work were treated in the masters thesis of second author, directed by the first. Thanks to Wilfried Imrich, whose work on prime factorization of nonbipartite graphs served as inspiration for many parts of this paper. Many thanks also to the referees for their careful reading and invaluable comments.

2. The Cartesian product

The *Cartesian product* of two graphs $A, B \in \Gamma$ is the graph $A \,\square\, B \in \Gamma$ with vertices $V(A) \times V(B)$ and edges

$$E(A \,\square\, B) = \big\{(a, b)(a', b') \mid aa' \in E(A) \text{ and } b = b', \text{ or } a = a' \text{ and } bb' \in E(B)\big\}.$$

(See Fig. 3.) The Cartesian product is commutative and associative in the sense that $A \,\square\, B \cong B \,\square\, A$ and $A \,\square\, (B \,\square\, C) \cong (A \,\square\, B) \,\square\, C$. Letting $B + C$ denote the disjoint union of graphs B and C, we also get the distributive law

$$A \,\square\, (B + C) = A \,\square\, B + A \,\square\, C, \tag{1}$$

which is true equality, rather than mere isomorphism.

Clearly $K_1 \,\square\, A \cong A$ for any graph A, so K_1 is the unit for the Cartesian product. A nontrivial graph G is *prime over* $\square$ if for any factoring $G \cong A \,\square\, B$, one of A or B is K_1 and the other is G. Certainly every graph can be factored into prime factors in Γ. Sabidussi and Vizing [7,8] proved that each connected graph has a unique prime factoring. More precisely, we have the following.

Theorem 1 (*Theorem 6.8 of [4]*)**.** *Let $G, H \in \Gamma$ be isomorphic connected graphs $G = G_1 \,\square\, \cdots \,\square\, G_k$ and $H = H_1 \,\square\, \cdots \,\square\, H_\ell$, where each factor G_i and H_i is prime. Then $k = \ell$, and for any isomorphism $\varphi : G \to H$, there is a permutation π of $\{1, 2, \ldots, k\}$ and isomorphisms $\varphi_i : G_{\pi(i)} \to H_i$ for which*

$$\varphi(x_1, x_2, \ldots, x_k) = \big(\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_k(x_{\pi(k)})\big).$$

Theorem 1 invites us to identify each H_i with $G_{\pi^{-1}(i)}$, yielding a corollary.

Corollary 1. *If $\varphi : G_1 \,\square\, \cdots \,\square\, G_k \to H_1 \,\square\, \cdots \,\square\, H_k$ is an isomorphism, and each G_i and H_i is prime, then the vertices of each H_i can be relabeled so that*

$$\varphi(x_1, x_2, \ldots, x_k) = (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)})$$

for a permutation π of $\{1, \ldots, k\}$. That is, φ merely permutes its arguments.

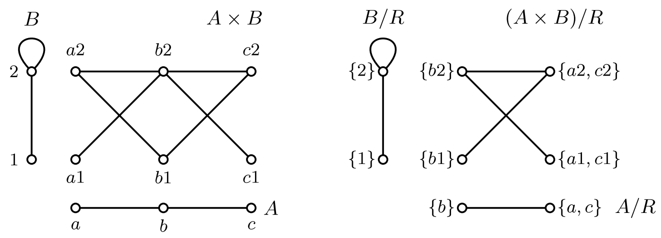


Fig. 4. Graphs A, B and $A \times C$ (left), and quotients A/R, B/R and $(A \times B)/R$ (right).

3. *R*-Thin graphs

The notion of so-called R-thinness is an important issue in factorings over the direct product. McKenzie [6] uses this idea (in a somewhat more general form), citing an earlier use by Chang [1]. A graph G is called *R-thin* if no two vertices have the same neighborhood, that is, if $N_G(x) = N_G(y)$ implies $x = y$. Said differently, any vertex is uniquely determined by its neighborhood. Maps between R-thin graphs are often conveniently rigid. For example, an isomorphism between R-thin bipartite graphs is completely determined by its effect on one partite set, because the neighborhoods of the other vertices are contained in that partite set. For another example, we note in passing that an analogue of Theorem 1 for the direct product holds for R-thin nonbipartite graphs. See Theorem 8.15 of [4].

More generally, one forms a relation R on the vertices of an arbitrary graph. Two vertices x and x' of a graph G are in *relation* R, written xRx', precisely if their open neighborhoods are identical, that is, if $N_G(x) = N_G(x')$. It is a simple matter to check that R is an equivalence relation on $V(G)$. (For example, the equivalence classes for $A \times B$ in Fig. 4 are $\{b1\}$, $\{b2\}$, $\{a1, c1\}$ and $\{a2, c2\}$; those of A are $\{a, c\}$ and $\{b\}$, and those of B are $\{1\}$ and $\{2\}$.) A graph is then R-thin if and only if each R-equivalence class contains exactly one vertex.

Given two R-equivalence classes X and Y (not necessarily distinct), it is easy to check that either every vertex in X is adjacent to every vertex in Y, or no vertex in X is adjacent to any in Y.

Given a graph G, we define a quotient graph G/R (in Γ_0) whose vertex set is the set of R-equivalence classes of G, and for which two classes are adjacent if they are joined by an edge of G. (And a single class carries a loop provided that an edge of G has both endpoints in that class.) Fig. 4 shows quotients A/R, B/R and $(A \times B)/R$. If G is R-thin, then $G/R \cong G$. An easy check confirms that G/R is R-thin for any $G \in \Gamma_0$.

Given $x \in V(G)$ let $[x] = \{x' \in V(G) \mid N_G(x') = N_G(x)\}$ denote the R-equivalence class containing x. As the relation R is defined entirely in terms of adjacencies, it is clear that given an isomorphism $\varphi : G \to H$ we have xRy in G if and only if $\varphi(x)R\varphi(y)$ in H. Thus φ maps R-equivalence classes of G to R-equivalence classes of H, and in particular $\varphi([x]) = [\varphi(x)]$. Thus any isomorphism $\varphi : G \to H$ induces an isomorphism $\widetilde{\varphi} : G/R \to H/R$ defined as $\widetilde{\varphi}([x]) = [\varphi(x)]$. But an isomorphism $\widetilde{\varphi} : G/R \to H/R$ does not necessarily imply that there is an isomorphism $\varphi : G \to H$. (Consider $G = P_3$ and $H = K_2$.) However, we do have the following result in this direction. The straightforward proof can be found in Section 8.2 of [4].

Proposition 1. *If there is an isomorphism $\widetilde{\varphi} : G/R \to H/R$ with the property that $|X| = |\widetilde{\varphi}(X)|$ for each $X \in V(G/R)$, then there is also an isomorphism $\varphi : G \to H$. (Such a φ can be obtained from $\widetilde{\varphi}$ by declaring that φ restricts to a bijection $X \to \widetilde{\varphi}(X)$ for each X.)*

Fig. 4 (right) suggests an isomorphism $\eta : (A \times B)/R \to A/R \times B/R$ given by $[(v, w)] \mapsto ([v], [w])$. Indeed, this is a general principle, as is proved in section 8.2 of [4]. Moreover, the figure suggests $[(v, w)] = [v] \times [w]$ (as sets). This too is true in general. It follows that $\big|[(v, w)]\big| = \big|[v]\big| \cdot \big|[w]\big|$, and that $A \times B$ is R-thin if and only if both A and B are.

Remark 1. Suppose there is an isomorphism $\varphi : A \times B \to A' \times B'$, given as $\varphi(a, b) = (\varphi_A(a, b), \varphi_B(a, b))$. The above discussion implies the following commutative diagram of induced isomorphisms.

$$\begin{array}{ccc} (A\times B)/R & \xrightarrow{\widetilde{\varphi}} & (A'\times B')/R \\ \eta\downarrow & & \downarrow\eta \\ A/R\times B/R & \xrightarrow{\widetilde{\varphi}} & A'/R\times B'/R \end{array} \qquad \begin{array}{ccc} [(a,b)] & \longmapsto & [(\varphi_A(a,b),\varphi_B(a,b))] \\ \downarrow & & \downarrow \\ \big([a],[b]\big) & \longmapsto & ([\varphi_A(a,b)]\,,[\varphi_B(a,b)]) \end{array}$$

Moreover,

$$\begin{aligned} |\,[(a,b)]\,| &= \left|\,\left[\big(\varphi_A(a,b),\varphi_B(a,b)\big)\right]\,\right| \\ = |[a]|\cdot|[b]| &= |\,[\varphi_A(a,b)]\,|\cdot|\,[\varphi_B(a,b)]\,|\,. \end{aligned}$$

The proof of our main theorem will use these facts often.

We have noted that a factoring $G \cong A \times B$ induces a corresponding factoring $G/R \cong A/R \times B/R$. In general the converse is false: For an arbitrary G, a factoring $G/R \cong A \times B$ cannot necessarily be "pulled up" to a factoring $G = A' \times B'$ for which $A'/R \cong A$ and $B'/R \cong B$. But there is a partial converse:

Proposition 2 (*Proposition 8.6 of [4]*)**.** *Suppose $G \in \Gamma_0$ has no isolated vertices, and $G/R \cong A \times B$. Thus there is a bijective correspondence between R-equivalence classes of G and vertices $(a, b) \in V(A \times B)$. Let $|(a, b)|$ stand for the cardinality of the R-equivalence class corresponding to (a, b). If there are maps $\alpha : V(A) \to \mathbb{N}$ and $\beta : V(B) \to \mathbb{N}$ for which $|(a, b)| = \alpha(a) \cdot \beta(b)$, then there is a factoring $G \cong A' \times B'$, where $A'/R \cong A$ and $B'/R \cong B$.*

In Proposition 2, A' is formed by "blowing up" each $a \in V(A)$ to an R-class with $\alpha(a)$ vertices; there is an edge between two vertices of A' precisely when A has an edge joining the corresponding vertices. The graph B' is formed similarly.

Let K_n^* be the complete graph on n vertices, with a loop at each vertex.

Corollary 2 (*Corollary 8.7 of [4]*)**.** *A graph $B \in \Gamma_0$ factors as $B = K_n^* \times H$ if and only if n divides the cardinality of each R-equivalence class of B. In particular, the sizes of the R-equivalence classes of a prime graph are relatively prime.*

Our main proof uses the ideas of this section freely, sometimes without comment. We assume our reader is well versed in this material. We refer the reader who requires proofs and explanations to Section 8.2 of [4].

4. The Cartesian skeleton

We now recall the definition of the Cartesian skeleton $S(G)$ of an arbitrary graph G in Γ_0. The Cartesian skeleton $S(G)$ is a graph on the vertex set of G that has the property $S(A \times B) = S(A) \,\square\, S(B)$ in the class of R-thin graphs, thereby linking the direct and Cartesian products.

We construct $S(G)$ as a certain subgraph of the Boolean square of G. The *Boolean square* of a graph G is the graph G^s with $V(G^s) = V(G)$ and $E(G^s) = \{xy \mid N_G(x) \cap N_G(y) \neq \emptyset\}$. Thus, xy is an edge of G^s whenever G has an x, y-walk of length two. For instance, if $p \geq 3$, then $K_p^s = K_p^*$ (i.e., K_p with a loop added to each vertex). Also, $K_2^s = K_1^* + K_1^*$ and $K_1^s = K_1$. The left side of Fig. 5 shows graphs A, B and $A \times B$ (bold) together with their Boolean squares A^s, B^s and $(A \times B)^s$ (dotted).

If G has an x, y-walk W of even length, then G^s has an x, y-walk of length $|W|/2$ on alternate vertices of W. Thus G^s is connected if G is connected and has an odd cycle. (An odd cycle guarantees an even walk between any two vertices of G.) On the other hand, if G is connected and bipartite, then G^s has exactly two components, and their respective vertex sets are the two partite sets of G.

We now show how to form $S(G)$ as a certain spanning subgraph of G^s. Consider an arbitrary factorization $G \cong A \times B$, by which we identify each vertex of G with an ordered pair (a, b). We say that an edge $(a, b)(a', b')$ of G^s is *Cartesian* relative to the factorization $A \times B$ if either $a = a'$ and $b \neq b'$, or $a \neq a'$ and $b = b'$. For example, in Fig. 5 edges xz and zy of G^s are Cartesian (relative to

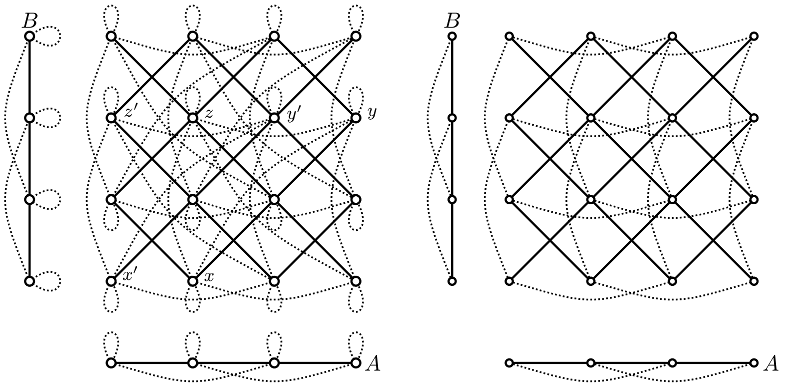


Fig. 5. Left: Graphs $A, B, A \times B$ and their Boolean squares A^s, B^s and $(A \times B)^s$ (dotted). Right: Graphs $A, B, A \times B$ and their Cartesian skeletons $S(A)$, $S(B)$ and $S(A \times B)$ (dotted).

the factorization $A \times B$), but edges xy and yy of G^s are not Cartesian. We will make $S(G)$ from G^s by removing the edges of G^s that are not Cartesian, but we do this in a way that does not reference the factoring $A \times B$ of G. We identify two intrinsic criteria for a non-loop edge of G^s that tell us if it may fail to be Cartesian relative to some factoring of G. (Note that the symbol $\subset$ means *proper* inclusion.)

(i) In Fig. 5 edge xy of G^s is not Cartesian, and there is a $z \in V(G)$ with $N_G(x) \cap N_G(y) \subset N_G(x) \cap N_G(z)$ and $N_G(x) \cap N_G(y) \subset N_G(y) \cap N_G(z)$.

(ii) In Fig. 5 the edge $x'y'$ of G^s is not Cartesian, and there is a $z' \in V(G)$ with $N_G(x') \subset N_G(z') \subset N_G(y')$.

Our aim is to remove from G^s all edges that meet one of these criteria. We package the above criteria into the following definition.

Definition 1. An edge xy of G^s is *dispensable* if $x = y$ or there exists $z \in V(G)$ for which both of the following statements hold.

(1) $N_G(x) \cap N_G(y) \subset N_G(x) \cap N_G(z)$ or $N_G(x) \subset N_G(z) \subset N_G(y)$,
(2) $N_G(y) \cap N_G(x) \subset N_G(y) \cap N_G(z)$ or $N_G(y) \subset N_G(z) \subset N_G(x)$.

Observe that the above statements (1) and (2) are symmetric in x and y. It is easy to confirm that (i) or (ii) holding for a triple x, y, z is equivalent to *both* of (1) and (2) holding. Now we come to the main definition of this section.

Definition 2. The *Cartesian skeleton* of a graph G is the spanning subgraph $S(G)$ of G^s obtained by removing all dispensable edges from G^s.

The right side of Fig. 5 is the same as its left side, except all dispensable edges of A^s, B^s and $(A \times B)^s$ are deleted. Thus the remaining dotted edges are $S(A)$, $S(B)$ and $S(A \times B)$. Note that although $S(G)$ was defined without regard to the factorization $G = A \times B$, we nonetheless have $S(A \times B) = S(A) \,\square\, S(B)$. The following proposition from [3] asserts that this always holds for R-thin graphs.

Proposition 3. *If A, B are R-thin graphs without isolated vertices, then $S(A \times B) = S(A) \,\square\, S(B)$. (This is equality, not mere isomorphism; the graphs $S(A \times B)$ and $S(A) \,\square\, S(B)$ have identical vertex and edge sets.)*

As $S(G)$ is defined entirely in terms of the adjacency structure of G, we have the following immediate consequence of Definition 2.

Proposition 4. *Any isomorphism $\varphi : G \to H$, as a map $V(G) \to V(H)$, is also an isomorphism $\varphi : S(G) \to S(H)$.*

We will also need a result concerning connectivity of Cartesian skeletons. The following result (which does not require R-thinness) is from [3]. (For another proof, see Chapter 8 of [4].)

Proposition 5. *Suppose G is connected.*

(i) *If G has an odd cycle, then $S(G)$ is connected.*
(ii) *If G is nontrivial bipartite, then $S(G)$ has two connected components. Their respective vertex sets are the two partite sets of G.*

5. Main result

Having discussed the necessary supporting material, we can now prove our main result.

Theorem 2. *Suppose a connected bipartite graph G factors as $G \cong A \times B$ and $G \cong A' \times B'$, where B and B' are prime and bipartite. Then $B \cong B'$.*

Proof. Let G be as stated. Take an isomorphism $\varphi : A \times B \to A' \times B'$. Then $\varphi(a, b) = (\varphi_A(a, b), \varphi_B(a, b))$ for certain component functions φ_A and φ_B.

We now reduce to R-thin graphs in order to apply the properties of the Cartesian skeleton operator. By Remark 1 in Section 3 there is an induced isomorphism

$$\begin{array}{lll} A/R \times B/R & \xrightarrow{\widetilde{\varphi}} & A'/R \times B'/R \\ ([a], [b]) & \mapsto & \big([\varphi_A(a, b)], [\varphi_B(a, b)]\big). \end{array} \tag{2}$$

For future reference, we note that Remark 1 of Section 3 also states

$$\big|[a]\big| \cdot \big|[b]\big| = \big|[\varphi_A(a, b)]\big| \cdot \big|[\varphi_B(a, b)]\big|. \tag{3}$$

From here the proof proceeds in four parts. Part 1 applies the Cartesian skeleton operator to the above map $\widetilde{\varphi}$ and employs unique factorization over the Cartesian product to uncover key structural properties of $\widetilde{\varphi}$. Part 2 uses this to obtain a factoring $B/R = S \times T$. Part 3 lifts this to a factoring $B = S' \times T'$ and uses primeness of B to deduce S and S' are trivial, greatly simplifying the situation. Part 4 uses these simplifications to get $B/R \cong B'/R$, and then lifts to an isomorphism $B \cong B'$.

Part 1. Applying the Cartesian skeleton operator and Proposition 4 to the above map (2), we see that $\widetilde{\varphi}$ is also an isomorphism

$$\widetilde{\varphi} : S(A/R \times B/R) \to S(A'/R \times B'/R).$$

This yields the upper-most square in the following Diagram (4). In this top square, both occurrences of $\widetilde{\varphi}$ are isomorphisms. Obviously, the vertical dotted arrows *do not* indicate isomorphisms, but they are the identity on vertex sets, and in this sense the top square of the diagram is commutative.

The remainder of the diagram proceeds as follows. Proposition 3 applied to the second line yields the third line. (The vertical double lines indicate equality and the horizontal arrows are isomorphisms.)

$$\begin{array}{ccc} A/R \times B/R & \xrightarrow{\qquad\widetilde{\varphi}\qquad} & A'/R \times B'/R \\ S \big\downarrow & & S \big\downarrow \\ S\big(A/R \times B/R\big) & \xrightarrow{\qquad\widetilde{\varphi}\qquad} & S\big(A'/R \times B'/R\big) \\ \| & & \| \\ S(A/R) \,\square\, S(B/R) & \xrightarrow{\qquad\widetilde{\varphi}\qquad} & S(A'/R) \,\square\, S(B'/R) \\ \| & & \| \\ S(A/R) \,\square\, (B_0 + B_1) & \xrightarrow{\qquad\widetilde{\varphi}\qquad} & S(A'/R) \,\square\, (B'_0 + B'_1) \\ \| & & \| \\ S(A/R) \,\square\, B_0 + S(A/R) \,\square\, B_1 & \xrightarrow{\qquad\widetilde{\varphi}\qquad} & S(A'/R) \,\square\, B'_0 + S(A'/R) \,\square\, B'_1. \end{array} \tag{4}$$

Concerning the fourth line, every R-equivalence class of B (i.e., every vertex of B/R) lies entirely in a partite set of B. Any edge of B/R runs from an R-equivalence class in one partite set of B to an R-equivalence class in the other partite set. It follows that B/R is bipartite, and it is connected because B is. Thus the Boolean square of B/R consists of two connected components whose respective vertex sets are the partite sets of B/R. In turn, the skeleton $S(B/R)$ consists of two connected components B_0 and B_1 whose respective vertex sets are the partite sets of B/R. (See Proposition 5.) Thus $S(B/R) = B_0 + B_1$. Similarly $S(B'/R) = B'_0 + B'_1$ for two other graphs B'_0 and B'_1. This gives the fourth line of Diagram (4). The bottom line follows from the distributive property of the Cartesian product.

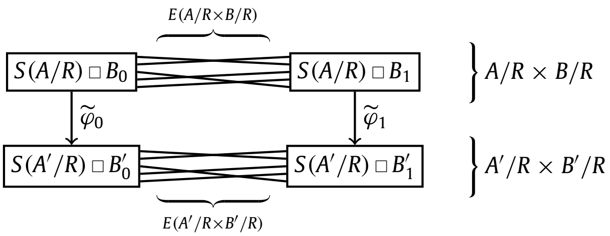
Consider the bottom line of Diagram (4). The isomorphism $\widetilde{\varphi}$ restricts to isomorphisms $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_1$ on the two components of $S(A/R)\,\square\,B_0 + S(A/R)\,\square\,B_1$. Arrange the indexing of their codomains as in Diagram (5), below.

$$\begin{array}{ccc} S(A/R)\,\square\,B_0 & \qquad & S(A/R)\,\square\,B_1 \\ \big\downarrow \widetilde{\varphi}_0 & & \big\downarrow \widetilde{\varphi}_1 \\ S(A'/R)\,\square\,B'_0 & & S(A'/R)\,\square\,B'_1 \end{array} \tag{5}$$

Let us reflect on Diagram (5). By Diagram (4), the graphs $A/R \times B/R$ and $S(A/R)\,\square\,B_0 + S(A/R)\,\square\,B_1$ have identical vertex sets. Moreover, the two partite sets of $A/R \times B/R$ are the vertices of $S(A/R)\,\square\,B_0$ and $S(A/R)\,\square\,B_1$. Similarly, $A'/R \times B'/R$ and $S(A'/R)\,\square\,B'_0 + S(A'/R)\,\square\,B'_1$ have identical vertex sets, and the two partite sets of $A'/R \times B'/R$ are the vertices of $S(A'/R)\,\square\,B'_0$ and $S(A'/R)\,\square\,B'_1$. The map $\widetilde{\varphi}$ is actually two isomorphisms

$$\widetilde{\varphi} : A/R \times B/R \to A'/R \times B'/R,$$
$$\widetilde{\varphi} : S(A/R)\,\square\,B_0 + S(A/R)\,\square\,B_1 \to S(A'/R)\,\square\,B'_0 + S(A'/R)\,\square\,B'_1,$$

where $\widetilde{\varphi}(x, y)$ is $\widetilde{\varphi}_0(x, y)$ or $\widetilde{\varphi}_1(x, y)$ depending on the partite set (respectively component) that (x, y) belongs to. We summarize this in the informal Diagram (6), a slight embellishment of Diagram (5).



(6)

With this in mind, we now refine Diagram (5). Identify $S(A/R)$ with its prime factorization

$$S(A/R) = A_1\,\square\,A_2\,\square\,\cdots\,\square\,A_i$$

over the Cartesian product. That is, we now label (or *coordinatize*) each vertex of $S(A/R)$ with the corresponding i-tuple $(a_1, a_2, \ldots, a_i)$ from its prime factorization. Similarly, form the prime factorizations

$$\begin{aligned} S(A'/R) &= A'_1\,\square\,A'_2\,\square\,\cdots\,\square\,A'_\ell, \\ B_0 &= B_{01}\,\square\,B_{02}\,\square\,\cdots\,\square\,B_{0j}, \\ B_1 &= B_{11}\,\square\,B_{12}\,\square\,\cdots\,\square\,B_{1k}, \\ B'_0 &= B'_{01}\,\square\,B'_{02}\,\square\,\cdots\,\square\,B'_{0m}, \\ B'_1 &= B'_{11}\,\square\,B'_{12}\,\square\,\cdots\,\square\,B'_{1n}. \end{aligned}$$

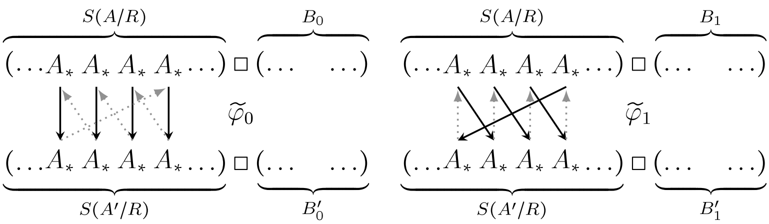


Fig. 6. An AA'-orbit. Solid arrows show effect of $\widetilde{\varphi}$. Dotted arrows on left show effect of $\widetilde{\varphi}_1^{-1}$ on $S(A/R)$. Dotted arrows on right show effect of $\widetilde{\varphi}_0^{-1}$ on $S(A/R)$. Vertical arrows are identity maps.

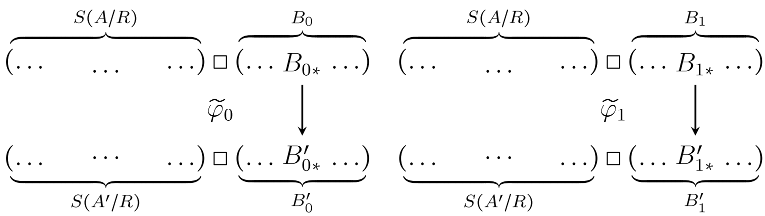


Fig. 7. A B_0B_0'-orbit on the left. A B_1B_1'-orbit on the right. Arrows are identity maps.

With these factorizations, Diagram (5) is updated to Diagram (7), below.

$$\begin{array}{ccc}
\overbrace{(A_1 \,\Box\, \cdots \,\Box\, A_i)}^{S(A/R)} \,\Box\, \overbrace{(B_{01} \,\Box\, \cdots \,\Box\, B_{0j})}^{B_0} & & \overbrace{(A_1 \,\Box\, \cdots \,\Box\, A_i)}^{S(A/R)} \,\Box\, \overbrace{(B_{11} \,\Box\, \cdots \,\Box\, B_{1k})}^{B_1} \\
\Big\downarrow \widetilde{\varphi}_0 & & \Big\downarrow \widetilde{\varphi}_1 \\
\underbrace{(A_1' \,\Box\, \cdots \,\Box\, A_\ell')}_{S(A'/R)} \,\Box\, \underbrace{(B_{01}' \,\Box\, \cdots \,\Box\, B_{0m}')}_{B_0'} & & \underbrace{(A_1' \,\Box\, \cdots \,\Box\, A_\ell')}_{S(A'/R)} \,\Box\, \underbrace{(B_{11}' \,\Box\, \cdots \,\Box\, B_{1n}')}_{B_1'}
\end{array} \qquad (7)$$

Concerning this diagram, Theorem 1 says each component function of $\widetilde{\varphi}_0$ is an isomorphism from a prime factor of $S(A/R) \,\Box\, B_0$ to a prime factor of $S(A'/R) \,\Box\, B_0'$. And each component function of $\widetilde{\varphi}_1$ is an isomorphism from a prime factor of $S(A/R) \,\Box\, B_1$ to one of $S(A'/R) \,\Box\, B_1'$. Together $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_1$ partition these factors into certain groups or *orbits*, as follows:

First consider factors of $S(A/R)$ that both $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_1$ send to factors of $S(A'/R)$, and that are related as in Fig. 6. Here $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_1^{-1}$, applied alternately and iteratively, permute the factors cyclically. (We use commutativity of $\Box$ to arrange the factors in a way that highlights this cyclic pattern.) Call such a configuration of factors an AA'-*orbit*. Note that any two factors in the same AA'-orbit are isomorphic. As in Corollary 1, let us identify domain with image for the solid vertical arrows on the left, so all vertical arrows are identity maps; but once this is done the solid arrows on the right are isomorphisms, not necessarily identity maps, as their images have already been relabeled to make the solid arrows on the left into identity maps.

Next, referring to Diagram (7), suppose there is a factor B_{0*} of B_0 that $\widetilde{\varphi}_0$ sends to a factor B_{0*}' of B_0'. (The asterisks represent definite but unspecified indices, generally different.) Call such a pair a B_0B_0'-*orbit*. The left side of Fig. 7 shows such an orbit. Unlike an AA'-orbit, this orbit terminates after one step because $\widetilde{\varphi}_1^{-1}$ does not apply to factors of B_0'. A similarly defined B_1B_1'-*orbit* is on the right. As in Corollary 1, we identify domain with image so the vertical arrows represent identity maps.

Now group together all factors of $S(A/R)$ that belong to AA'-orbits, and call their product K. Group together the factors of B_0 that belong to B_0B_0'-orbits and call their product P. Group together the factors of B_1 that belong to B_1B_1'-orbits and call their product Q. On the factors thus far considered, $\widetilde{\varphi}$ has the structure illustrated in Diagram (8). Vertical arrows represent identity maps, except for the one labeled

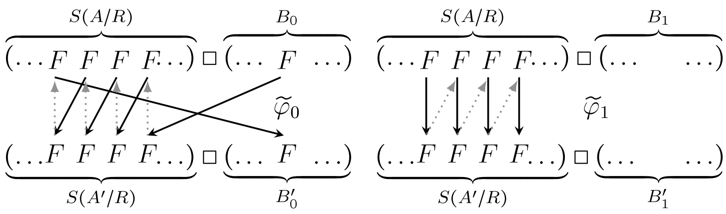


Fig. 8. An odd B_0B_0'-orbit of length 9. Arrows are identity maps. Solid arrows show effect of $\widetilde{\varphi}$. Dotted arrows on left show effect of $\widetilde{\varphi}_1^{-1}$. Dotted arrows on right show $\widetilde{\varphi}_0^{-1}$.

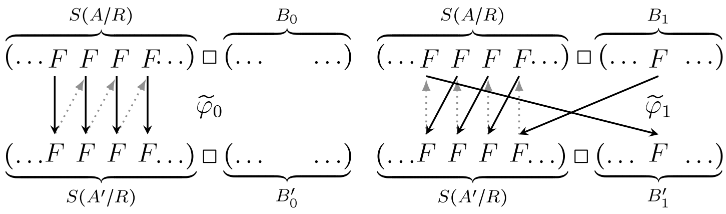
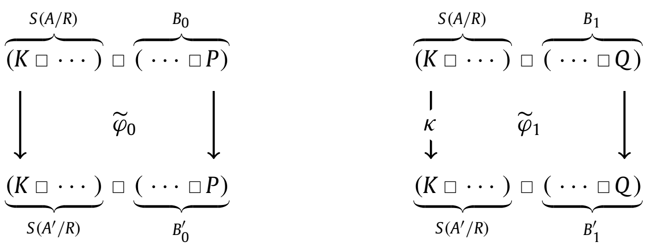


Fig. 9. An odd B_1B_1'-orbit of length 9. Solid arrows show effect of $\widetilde{\varphi}$. Dotted arrows on left show effect of $\widetilde{\varphi}_1^{-1}$. Dotted arrows on right show effect of $\widetilde{\varphi}_0^{-1}$. All arrows are identity maps.

κ, which is an isomorphism.



(8)

If it happened that P and Q accounted for all factors of B_0, B_0', B_1 and B_1', then we would be nearly done because the vertical maps $P \to P$ and $Q \to Q$ in Diagram (8) would yield an isomorphism $B/R \to B'/R$. In essence this will be our approach, but there are other orbits to examine. Consider Fig. 8, where $\widetilde{\varphi}_0$ sends a prime factor F of B_0 to prime factor F of $S(A'/R)$. Then $\widetilde{\varphi}_1^{-1}$ in turn sends this to a factor F of $S(A/R)$. Following $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_0^{-1}$ alternately, we eventually get to a factor F of B_0'. (Commutativity of $\square$ allows the arrangement of factors in the indicated pattern.) By identifying, in succession, each domain with its image in the zig-zag sequence on the left, all arrows in Fig. 8 are identity maps. We call such a configuration an *odd* B_0B_0'*-orbit* because it begins in B_0, ends in B_0', and the sequence of identity maps has an odd number of steps. The number of steps is called the *length* of the B_0B_0'-orbit. Note that the length of an odd B_0B_0'-orbit is twice its number of factors in $S(A/R)$, plus 1. The factor F in B_0 is called the *initial factor* of the orbit and the F in B_0' is the *terminal factor*.

Similarly, there may be orbits of the type shown in Fig. 9, which we call *odd* B_1B_1'*-orbits*. Here the initial factor F in B_1 is "pushed through" to the terminal factor F of B_1' when we apply $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_0^{-1}$ alternately. Identifying domains with images, in sequence, we take all arrows to be identity maps. As in the previous case, the orbit length is twice the number of factors in $S(A/R)$ (or $S(A'/R)$), plus 1.

We regard a B_0B_0'-orbit such as the one in Fig. 7 as an odd B_0B_0'-orbit of length 1. The B_1B_1'-orbit in Fig. 7 is an odd B_1B_1'-orbit of length 1. Note that any odd orbit has an initial factor F in B_0 (or B_1) and a "twin" terminal factor F in B_0' (or B_1').

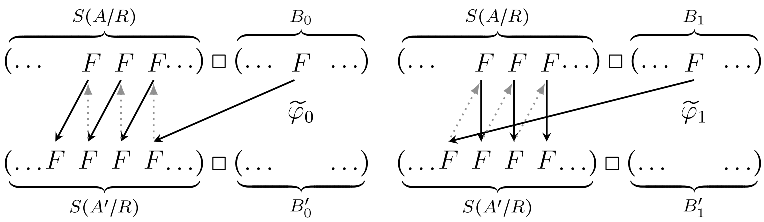


Fig. 10. An even B_0B_1-orbit of length 8. Solid arrows show effect of $\widetilde{\varphi}$. Dotted arrows on left show effect of $\widetilde{\varphi}_1^{-1}$. Dotted arrows on right show effect of $\widetilde{\varphi}_0^{-1}$. All arrows are identity maps.

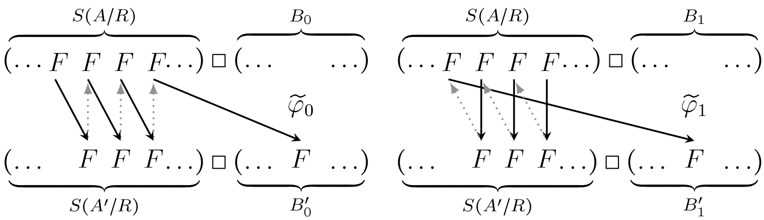
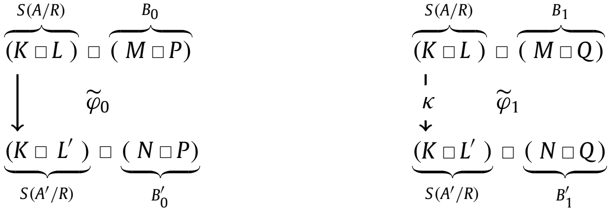


Fig. 11. An even $B_0'B_1'$-orbit of length 8. Arrows represent identity functions.

What other kinds of orbits are there? A moment's reflection reveals that there are only two more possibilities. One is illustrated in Fig. 10. Let us call this an *even* B_0B_1*-orbit* because alternatively applying $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_1^{-1}$ an even number of times sends a factor F of B_0 to a factor F of B_1.

The final type of orbit is illustrated in Fig. 11. It is called an *even* $B_0'B_1'$*-orbit*, because alternatively applying $\widetilde{\varphi}_0^{-1}$ and $\widetilde{\varphi}_1$ an even number of times sends a factor F of B_0' to a factor F of B_1'. Notice that any even orbit has twin initial and terminal factors in B_0 and B_1 (or in B_0' and B_1'). Identifying domains with images, all arrows represent identity maps.

Let us update Diagram (8) to reflect this discussion of even and odd orbits. In our current parlance, the P in Diagram (8) is the product of prime factors of B_0 (or B_0') that belong to odd orbits *of length 1*. And the Q is the product of prime factors of B_1 (or B_1') that belong to odd orbits *of length 1*. Let us modify this so that P is the product of all prime factors of B_0 (or B_0') that belong to *odd* orbits of *any length*. Likewise Q is now the product of prime factors of B_1 (or B_1') that belong to odd orbits of any length. With this, Diagram (8) is updated to the following Diagram (9). Here M is the product of prime factors of B_0 (or B_1) that belong to even B_0B_1-orbits, and N is the product of prime factors of B_0' (or B_1') that belong to even $B_0'B_1'$-orbits. Also L is the product of factors of $S(A/R)$ that are not accounted for in K, that is, L is the product of all factors of $S(A/R)$ that belong to even or odd orbits. Finally L' is the product of factors of $S(A'/R)$ that are not accounted for in K, that is, it is the product of all factors of $S(A'/R)$ that belong to even or odd orbits. Notice $L = L'$ if there are no even orbits.



(9)

Now we codify the structure of $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_1$ relative to Diagram (9). By the structures of the orbits, we have

$$\widetilde{\varphi}_0\big((k,\ell),(m,p)\big) = \big((k,\quad \lambda_0(\ell,m,p)),\ (\mu_0(\ell),\pi(\ell,p))\big) \tag{10}$$

$$\widetilde{\varphi}_1\big((k,\ell),(m,q)\big) = \big((\kappa(k),\lambda_1(\ell,m,q)),\ (\mu_1(\ell),\chi(\ell,q))\big) \tag{11}$$

for certain component functions $\lambda_0, \lambda_1, \mu_0, \mu_1, \pi$ and χ. Although we will not describe all these functions explicitly, it is not hard to see how they operate. For example, $\lambda_0(\ell, m, p) \in V(L')$ is a certain "shifting" of $\ell \in V(L)$. It shifts each coordinate of an odd B_0B_0'-orbit in ℓ to the left (as illustrated in Fig. 8) inserting a coordinate of p into the vacated spot. The coordinate in ℓ corresponding to the factor of this orbit that $\widetilde{\varphi}_0$ sends to B_0' shows up in $\pi(\ell, p) \in V(P)$. (Note that π includes p as an argument because of the possible presence of odd B_0B_0' orbits of length 1, as in Fig. 7.) Likewise, $\widetilde{\varphi}_0$ shifts each even B_0B_1-orbit to the left (as in Fig. 10), incorporating a factor of M, so on this orbit $\lambda_0(\ell, m, p)$ is a corresponding shifting of ℓ, with a coordinate of m inserted. Moreover, $\widetilde{\varphi}_0$ sends a factor of L in each even $B_0'B_1'$-orbit to a factor of N, and the coordinates in ℓ corresponding these factors show up in $\mu_0(\ell) \in V(N)$.

Actually, we will not need explicit descriptions of these component functions; the main point is the dependencies among factors expressed in Eqs. (10) and (11).

Observe that, by Eq. (11), the inverse of $\widetilde{\varphi}_1$, has the structure

$$\widetilde{\varphi}_1^{-1}\big((k, \ell), (n, q)\big) = \big((\kappa^{-1}(k), \lambda_1'(\ell, n, q)),\ (\mu_1'(\ell), \chi'(\ell, q))\big). \tag{12}$$

Notice that Diagram (9) has lost the identity maps $P \to P$ and $Q \to Q$ that were present in Diagram (8), but we will soon overcome this with a "merge operation" that will be defined in Part 2. Then we will see that primeness of B and B' implies that there are no even B_0B_1-orbits, and no even $B_0'B_1'$-orbits, which will mean $M = N = K_1$ and $L = L'$, greatly simplifying the situation.

Some remarks on notation. Our graphs now have several coordinatizations. A vertex of $A/R \times B/R$ could be written as (a, b), or $((k, l), (m, p))$ or even with some finer factorization involving orbits. At any point in the remainder of the proof we will use whichever coordinatization is most convenient, or clearest. Also, wherever possible we exercise typographical consistency, so that we will know, say, that a vertex (m, p) belongs to $M \,\square\, P = B_0$, etc. Moreover, for any edge bb' of B/R or B'/R it is understood that the left endpoint always belongs to the partite set $V(B_0)$ (or $V(B_0')$) and the right to $V(B_1)$ (or $V(B_1')$). Likewise, if $(a, b)(a', b') \in A/R \times B/R$ then the left vertex (a, b) belongs to $S(A/R) \,\square\, B_0$, and (a', b') to $S(A/R) \,\square\, B_1$, etc.

Part 2. Recall that the graphs $S(B/R)$ and $S(B'/R)$ are disjoint unions $B_0 + B_1$ and $B_0' + B_1'$, respectively. By Diagram (9), these graphs factor as

$$S(B/R) = M \,\square\, P + M \,\square\, Q = M \,\square\, (P + Q)$$
$$S(B'/R) = N \,\square\, P + N \,\square\, Q = N \,\square\, (P + Q).$$

This part of the proof shows that the above factoring of $S(B/R)$ induces a factoring $B/R = S \times T$. It will be clear that an exactly analogous argument gives a similar factoring of B'/R.

Define S as $V(S) = V(M)$ and $E(S) = \{m_0m_1 \mid (m_0, p)(m_1, q) \in E(B/R)\}$. Let T be the bipartite graph whose partite sets are the disjoint union $V(T) = V(P) \cup V(Q)$, and $E(T) = \{pq \mid (m_0, p)(m_1, q) \in E(B/R)\}$. Clearly, then, each edge $(m_0, p)(m_1, q)$ of B/R also belongs to $S \times T$, so $B/R \subseteq S \times T$.

The reverse inclusion takes work. Suppose $(m_0, p)(m_1, q) \in E(S \times T)$. Then $m_0m_1 \in E(S)$ and $pq \in E(T)$. By construction of S and T, this means that B/R has edges $(m_0, p')(m_1, q')$ and $(m_0', p)(m_1', q)$. We need to merge these into an edge $(m_0, p)(m_1, q) \in B/R$. The remainder of Part 2 is devoted to this goal. Once it is done we will have $B/R = S \times T$.

To reach our goal, we define a so-called *merge operation* that will, when applied iteratively, push edges of B/R, through orbits, to edges of B'/R. This operation is a sequence of six steps. It operates on an "input" edge $(a_0, c_0)(a_1, c_1) \in E(A/R \times B/R)$, blending in information from fixed edges $b_0b_1 \in E(B/R)$ and $b_0'b_1' \in E(B'/R)$.

Merge Operation with edges $b_0b_1 \in E(B/R)$ and $b_0'b_1' \in E(B'/R)$

0.	Initial step. Begin with input edge	$(a_0, c_0)(a_1, c_1) \in E(A/R \times B/R)$
1.	Interchange factors in A/R	$(a_1, c_0)(a_0, c_1) \in E(A/R \times B/R)$
2.	Insert edge b_0b_1 into second factor	$(a_1, b_0)(a_0, b_1) \in E(A/R \times B/R)$
3.	Apply $\widetilde{\varphi}$ to each endpoint	$(a_2, c_2)(a_3, c_3) \in E(A'/R \times B'/R)$
4.	Interchange factors in A'/R	$(a_3, c_2)(a_2, c_3) \in E(A'/R \times B'/R)$
5.	Insert edge $b_0'b_1'$ into second factor	$(a_3, b_0')(a_2, b_1') \in E(A'/R \times B'/R)$
6.	Apply $\widetilde{\varphi}^{-1}$ to each endpoint	$(a_4, c_4)(a_5, c_5) \in E(A/R \times B/R)$

This operation can be applied iteratively; the output $(a_4, c_4)(a_5, c_5)$ can be placed in line 0, and the process repeated.

We are now going to examine how the merge operation acts on orbits. For the remainder of Part 2, the edges $b_0b_1 \in E(B/R)$ and $b'_0b'_1 \in E(B'/R)$ are fixed (but arbitrary).

Let us see how the merge operation affects an odd $B_0B'_0$-orbit of length $2k+1$. (Consult Fig. 8 for the structure of such an orbit, and for how $\widetilde{\varphi}$ acts on it.) Consider an initial input edge

$$(a_0, c_0)(a_1, c_1) = \big((-\, d_1, d_2, \dots, d_k\, -), (-\, d_{k+1}\, -)\big)\ \big((-\, e_k, e_{k-1}, \dots, e_1\, -), (-)\big)$$

of $A/R \times B/R$, where for clarity we have listed only the coordinates in the orbit. The dashes indicate the presence of factors that we are ignoring; the choice of indexing will become apparent shortly. Say $b_0b_1 = (-\, f\, -)(-)$ and $b'_0b'_1 = (-\, f'\, -)(-)$, where the dashes again represent factors that are not in the orbit. (The orbit has no factors in B_1, so we write $b_1 = (-)$, and similarly for b'_1.) The merge operation transforms this edge as follows.

$$\begin{array}{lll}
0. & \big((-\, d_1,\ d_2, \dots, d_{k-1},\ d_k -), (-d_{k+1}-)\big) & \big((-\ e_k, e_{k-1}, \dots, e_2,\ \ e_1\, -), (-)\big)\\
1. & \big((-\, e_k,\ e_{k-1}, \dots, e_2,\ \ e_1 -), (-d_{k+1}-)\big) & \big((-\ d_1,\ d_2, \dots, d_{k-1}, d_k\, -), (-)\big)\\
2. & \big((-\, e_k,\ e_{k-1}, \dots, e_2,\ \ e_1\, -), (-\, f\, -)\big) & \big((-\ d_1,\ d_2, \dots, d_{k-1}, d_k\, -), (-)\big)\\
3. & \big((-\, e_{k-1}, e_{k-2}, \dots, e_1,\ f\ -), (-\, e_k\, -)\big) & \big((-\ d_1,\ d_2, \dots, d_{k-1}, d_k\, -), (-)\big)\\
4. & \big((-\, d_1,\ d_2, \dots, d_{k-1},\ \ d_k\ -), (-\, e_k\, -)\big) & \big((\, -e_{k-1},\ e_{k-2}, \dots, e_1, f\, -), (-)\big)\\
5. & \big((-\, d_1,\ d_2, \dots, d_{k-1},\ d_k\ -), (-\, f'\, -)\big) & \big((-\, e_{k-1},\ e_{k-2}, \dots, e_1, f\, -), (-)\big)\\
6. & \big((-\, f', d_1, \dots, d_{k-2}, d_{k-1}\, -), (-\, d_k\, -)\big) & \big((-\, e_{k-1},\ e_{k-2}, \dots, e_1, f\, -), (-)\big).
\end{array}$$

In summary, the merge operation transforms the edge

$$\big((-\, d_1, d_2, \dots, d_k\, -), (-\, d_{k+1}\, -)\big)\ \big((-\, e_k, e_{k-1}, \dots, e_1\, -), (-)\big) \in E(A/R \times B/R)$$

to the edge

$$\big((-\, f', d_1, d_2 \dots, d_{k-1}\, -), (-\, d_k\, -)\big)\ \big((-\, e_{k-1}, e_{k-2}, \dots, e_1, f\, -), (-)\big)$$

of $A/R \times B/R$. The net effect is to insert an f' on the far left and shift the remaining coordinates of the left endpoint one space to the right, and to insert an f on the far right and shift the coordinates of the right endpoint one space to the left. The d_{k+1} and the e_k are discarded. And if we apply the merge operation again to the edge just obtained, we get the following edge in $A/R \times B/R$, with d_k and e_{k-1} discarded.

$$\big((-\, f', f', d_1, \dots, d_{k-2}\, -), (-\, d_{k-1}\, -)\big)\ \big((-\, e_{k-2}, , \dots, e_1, f, f\, -), (-)\big).$$

To this, apply the merge operation again, for a total of k iterations to get

$$\big((-\, f', f', \dots, f', f'\, -), (-\, d_1\, -)\big)\ \big((-\, f, f, \dots, f, f\, -), (-)\big) \in E(A/R \times B/R).$$

One more iteration yields

$$\big((-\, f', f', \dots, f', f'\, -), (-\, f'\, -)\big)\ \big((-\, f, f, \dots, f, f\, -), (-)\big) \in E(A/R \times B/R).$$

After this we may apply the merge operation any number of times, but it causes no further changes on this orbit. The orbit is now packed with coordinates from b_0 and b'_0.

We above focused on a single orbit, but the merge operation affects all orbits simultaneously. For example, take an odd $B_1B'_1$-orbit of length $2\ell + 1$:

$$\big((-\, e_\ell, e_{\ell-1}, \dots, e_1\, -), (-)\big)\ \big((-\, d_1, d_2, \dots, d_\ell\, -), (-\, d_{\ell+1}\, -)\big) \in E(A/R \times B/R).$$

Ignoring coordinates not in this orbit, we have $b_0b_1 = (-)(-\ g\ -) \in E(B/R)$ and $b'_0b'_1 = (-)(-\, g'\, -) \in E(B'/R)$. After $\ell + 1$ (or more) iterations the merge operation produces

$$\big((-\, g, g, \dots, g\, -), (-)\big)\ \big((-\, g', g', \dots, g'\, -), (-\, g'\, -)\big) \in E(A/R \times B/R),$$

as the reader may check.

The key point is this: Suppose that $b_0b_1 = (-\ f\ -)(-\ g\ -) \in E(B/R)$ and $b_0'b_1' = (-\ f'\ -)(-\ g'\ -) \in E(B'/R)$, where f, f' belong to the same odd B_0B_0'-orbit of length $2k+1$, and g, g' belong to the same odd B_1B_1'-orbit of length $2\ell+1$. After at least $\max\{k, \ell\}+1$ iterations the merge operation gives an edge

$$((-f', \dots, f'\ -\ g, \dots, g\ -), (-f'\ -))((-f, \dots, f\ -\ g', \dots, g'-), (-\ g'\ -))$$

of $A/R \times B/R$. So although we started with $b_0'b_1' = (-\ f'\ -)(-\ g'\ -) \in E(B'/R)$, we get an edge $(-\ f'\ -)(-\ g'\ -) \in E(B/R)$ that equals $b_0'b_1'$ on the orbits being considered.

Having seen what the merge operation does to individual odd orbits, we can understand its effect on all odd orbits simultaneously. Recall $S(B/R) = M \square P + M \square Q$, where P is the product of the factors of odd orbits in B_0 and Q is the product of the factors of odd orbits in B_1. Likewise $S(B/R) = N \square P + N \square Q$. Say $b_0b_1 = (m_0, p)(m_1, q)$ and $b_0'b_1' = (n_0, p')(n_1, q')$. Apply the merge operation a number of times that exceeds the length of all odd orbits. The previous paragraph implies that the result is an edge $((-), (m_0', p'))((-), (m_1', q')) \in E(A/R \times B/R)$. Consequently there is an edge $(m_0', p')(m_1', q') \in E(B/R)$. We summarize this in a remark.

Remark 1. Suppose $(m_0, p)(m_1, q) \in E(B/R)$. Applying the merge operation with this edge and $(n_0, p')(n_1, q') \in E(B'/R)$ (as was done above) a sufficient number of times yields an edge of form $(m_0', p')(m_1', q') \in E(B/R)$. In particular, for any $(n_0, p')(n_1, q') \in E(B'/R)$, there exists some edge $(m_0, p')(m_1, q') \in E(B/R)$. Analogous reasoning yields a converse: Given $(m_0, p)(m_1, q) \in E(B/R)$ there is an edge $(n_0, p)(n_1, q) \in E(B'/R)$.

Next we examine the effects of the merge operation on *even* B_0B_1-orbits. To simplify the indexing, let us look at such an orbit of length 8, like the one illustrated in Fig. 10. The general picture should be clear from this example. We begin with an input edge

$$\big((-\ d_1, d_2, d_3\ -), (-\ d_4\ -)\big)\ \big((-\ e_3, e_2, e_1\ -), (-\ e_4\ -)\big) \in E(A/R \times B/R),$$

where we ignore coordinates not in the orbit. We write $b_0b_1 \in E(B/R)$ as $b_0b_1 = (-\ f_0\ -)(-\ f_1\ -)$, listing only the coordinates in the orbit. Edge $b_0'b_1' \in E(B/R)$ has no coordinates in the orbit, so we write $b_0'b_1' = (-)(-)$.

The first iteration of the merge operation is as follows.

$$\begin{array}{ll}
0. & \big((--\ d_1, d_2, d_3\ -), (-\ d_4\ -)\big)\ \big((--\ e_3, e_2, e_1\ -), (-\ e_4\ -)\big) \in E(A/R \times B/R)\\
1. & \big((--\ e_3, e_2, e_1\ -), (-\ d_4\ -)\big)\ \big((--\ d_1, d_2, d_3\ -), (-\ e_4\ -)\big) \in E(A/R \times B/R)\\
2. & \big((--\ e_3, e_2, e_1\ -), (-f_0-)\big)\ \big((--\ d_1, d_2, d_3\ -), (-f_1-)\big) \in E(A/R \times B/R)\\
3. & \big((-\ e_3, e_2, e_1, f_0\ -), (---)\big)\ \big((-f_1, d_1, d_2, d_3\ -), (---)\big) \in E(A'/R \times B'/R)\\
4. & \big((-f_1, d_1, d_2, d_3\ -), (---)\big)\ \big((-\ e_3, e_2, e_1, f_0\ -), (---)\big) \in E(A'/R \times B'/R)\\
5. & \big((-f_1, d_1, d_2, d_3\ -), (---)\big)\ \big((-\ e_3, e_2, e_1, f_0\ -), (---)\big) \in E(A'/R \times B'/R)\\
6. & \big((--\ f_1, d_1, d_2\ -), (-d_3-)\big)\ \big((--\ e_2, e_1, f_0\ -), (-\ e_3-)\big) \in E(A/R \times B/R).
\end{array}$$

Thereafter the second, third and fourth iterations give

$$\begin{array}{l}
\big((-f_1, f_1, d_1\ -), (-\ d_2\ -)\big)\ \big((-\ e_1, f_0, f_0\ -), (-\ e_2\ -)\big) \in E(A/R \times B/R),\\
\big((-f_1, f_1, f_1\ -), (-\ d_1\ -)\big)\ \big((-\ f_0, f_0, f_0\ -), (-\ e_1\ -)\big) \in E(A/R \times B/R),\\
\big((-f_1, f_1, f_1\ -), (-\ f_1\ -)\big)\ \big((-\ f_0, f_0, f_0\ -), (-\ f_0\ -)\big) \in E(A/R \times B/R),
\end{array}$$

and thereafter any further iteration produce no changes. In general, if the orbit had length k then we reach this stable state after k iterations. If the number of iterations exceeds the length of the longest orbit, then we have reached such a stable state on all orbits.

Notice that we began the merge with $(-f_0-)(-f_1-) \in E(B/R)$ and ended with $(-f_1-)(-f_0-) \in E(B/R)$. Such is true regardless of which even orbit we are dealing with. As $S(B/R) = M \square P + M \square Q$, where M is the product of the factors of even orbits, this gives the following:

Remark 2. Suppose $(m_0, p)(m_1, q) \in E(B/R)$. Applying the merge operation with this edge and $(n_0, p')(n_1, q') \in E(B'/R)$ (as was done above) a sufficient number of times yields $(m_1, p'')(m_0, q'') \in E(B/R)$. In fact, by Remark 1, we have precisely $(m_1, p')(m_0, q') \in E(B/R)$. To summarize, if $(m_0, p)(m_1, q) \in E(B/R)$ and $(n_0, p')(n_1, q') \in E(B'/R)$, then $(m_1, p')(m_0, q') \in E(B/R)$.

Finally we can finish our demonstration that $B/R = S \times T$. It remains to show $S \times T \subseteq B/R$. Suppose $(m_0, p)(m_1, q) \in S \times T$. By definition of S and T there are edges $(m_0, p')(m_1, q')$ and $(m_0', p)(m_1', q)$ in B/R. By Remark 1 there is some $(n_0, p)(n_1, q) \in E(B'/R)$. Applying Remark 2 to $(m_0, p')(m_1, q') \in E(B/R)$ and $(n_0, p)(n_1, q) \in E(B'/R)$ it follows that $(m_1, p)(m_0, q) \in E(B/R)$. Applying Remark 2 to $(m_1, p)(m_0, q) \in E(B/R)$ and $(n_0, p)(n_1, q) \in E(B'/R)$ we get $(m_0, p)(m_1, q) \in E(B/R)$. Therefore $S \times T \subseteq B/R$ and hence $B/R = S \times T$.

Part 3. Next we show that the factoring $B/R = S \times T$ lifts to a factoring $B = S' \times T'$ with $S'/R = S$ and $T'/R = T$. Then, because the bipartite graph T' is nontrivial we will have $S' = K_1$ and we will find $S = K_1$. As $M = V(S)$ we will get $M = K_1$ as well.

Keep in mind that each vertex (m, x) of $B/R = S \times T$ is an R-equivalence class of B, so it has a cardinality $|(m, x)|$. By Proposition 2 of Section 3, to show that $B = S' \times T'$ we just need to produce functions $\alpha : V(S) \to \mathbb{N}$ and $\beta : V(T) \to \mathbb{N}$ for which $|(m, x)| = \alpha(m) \cdot \beta(x)$. By definition, $V(T) = V(P) \cup V(Q)$, so we want $|(m, p)| = \alpha(m) \cdot \beta(p)$ and $|(m, q)| = \alpha(m) \cdot \beta(q)$. Let (m, p) and (m, q) be two vertices in the two partite sets of B/R with the same first coordinate m. We now build α and β.

Fix a vertex $k_0 \in V(K)$. Let (k_0, ℓ) be the vertex of $A/R = K \,\square\, L$ for which ℓ is chosen so that each coordinate in ℓ in a given orbit equals that orbit's coordinate in (m, p) or (m, q). For example, given an odd B_0B_0'-orbit, which has a factor in P, we have $(m, p) = (m, - f -)$ where f is the coordinate of the orbit's factor in P. Then, listing only its coordinates in the orbit, we put $\ell = (- f, f, \dots, f -)$. Similarly, for a B_1B_1'-orbit, which has a factor in Q, we have $(m, q) = (m, - e -)$, so $\ell = (- e, e, \dots, e -)$ on this orbit. And for an (even) B_0B_1-orbit, which has a factor in M, we have $(m, p) = (- g -, p)$ and $(m, q) = (- g -, q)$, so we put $\ell = (- g, g, \dots, g -)$ on this orbit. Finally, any even $B_0'B_1'$-orbit has no factors in M, P nor Q, but it has factors in L. On such an orbit just put $\ell = (- h, h, \dots, h -)$, for some h.

From the choice of ℓ and by the way $\widetilde{\varphi}$ acts on the orbits, it follows that

$$\widetilde{\varphi}_0\big((k_0, \ell), (m, p)\big) = \big((k_0, \ell'), (n, p)\big),$$
$$\widetilde{\varphi}_1\big((k_0, \ell), (m, q)\big) = \big((\kappa(k_0), \ell'), (n, q)\big)$$

for some $\ell' \in L'$ and $n \in N$. Applying Eq. (3) to these produces

$$|(k_0, \ell)| \cdot |(m, p)| = |(k_0, \ell')| \cdot |(n, p)|,$$
$$|(k_0, \ell)| \cdot |(m, q)| = |(\kappa(k_0), \ell')| \cdot |(n, q)|.$$

Dividing the first by the second yields

$$\frac{|(m, p)|}{|(m, q)|} = \frac{|(k_0, \ell')|}{|(\kappa(k_0), \ell')|} \cdot \frac{|(n, p)|}{|(n, q)|}. \tag{13}$$

Our functions α and β will come from this equation, but first we show that it simplifies in the sense that $\frac{|(k_0, \ell')|}{|(\kappa(k_0), \ell')|}$ is a constant. To deduce this we will use an iterative process to show that a quotient $\frac{|(k_0, \ell_0')|}{|(\kappa(k_0), \ell_0')|}$ does not depend on ℓ_0'. Fix vertices (m_0, p_0) and (n_0, q_0) of $M \,\square\, P$ and $N \,\square\, Q$. For given $\ell_i' \in V(L')$ Eq. (12) says that $\widetilde{\varphi}_1^{-1}\big((k_0, \ell_i'), (n_0, q_0)\big) = \big((\kappa^{-1}(k_0), \ell_{i+1}), (n_{i+1}, q_{i+1})\big)$ for some $\ell_{i+1} \in V(L)$, $n_{i+1} \in V(N)$ and $q_{i+1} \in V(Q)$. Eq. (12) again implies that for any k,

$$\widetilde{\varphi}_1^{-1}\big((k, \ell_i'), (n_0, q_0)\big) = \big((\kappa^{-1}(k), \ell_{i+1}), (n_{i+1}, q_{i+1})\big), \tag{14}$$

and applying Eq. (3),

$$|(k, \ell_i')| \cdot |(n_0, q_0)| = |(\kappa^{-1}(k), \ell_{i+1})| \cdot |(n_{i+1}, q_{i+1})|. \tag{15}$$

In turn, Eq. (10) says $\widetilde{\varphi}_0((k_0, \ell_{i+1}), (m_0, p_0)) = ((k_0, \ell_{i+2}'), (m_{i+2}, p_{i+2}))$ for some $\ell_{i+2}' \in V(L')$, $m_{i+2} \in V(M)$ and $p_{i+2} \in V(P)$. Then in fact for any k, Eq. (10) says

$$\widetilde{\varphi}_0\big((k, \ell_{i+1}), (m_0, p_0)\big) = \big((k, \ell_{i+2}'), (m_{i+2}, p_{i+2})\big). \tag{16}$$

Applying Eq. (3) to this, we see that for any k,

$$|(k,\ell_{i+1})|\cdot|(m_0,p_0)| = |(k,\ell'_{i+2})|\cdot|(m_{i+2},p_{i+2})|. \tag{17}$$

Using the set-up in this paragraph, consider the following sequence of steps.

$$\begin{aligned}
1.\quad \frac{|(k_0,\ell'_i)|}{|(\kappa(k_0),\ell'_i)|} &= \frac{|(k_0,\ell'_i)|\ \cdot\ |(n_0,q_0)|}{|(\kappa(k_0),\ell'_i)|\cdot|(n_0,q_0)|} \quad \text{(Insert } (n_0,q_0))\\
2.\quad &= \frac{|(\kappa^{-1}(k_0),\ell_{i+1})|\cdot|(n_{i+1},q_{i+1})|}{|(k_0,\ell_{i+1})|\ \cdot\ |(n_{i+1},q_{i+1})|} \quad \text{(By Eq. (15))}\\
3.\quad &= \frac{|(\kappa^{-1}(k_0),\ell_{i+1})|}{|(k_0,\ell_{i+1})|}\\
4.\quad &= \frac{|(\kappa^{-1}(k_0),\ell_{i+1})|\cdot|(m_0,p_0)|}{|(k_0,\ell_{i+1})|\ \cdot\ |(m_0,p_0)|} \quad \text{(Insert } (m_0,p_0))\\
5.\quad &= \frac{|(\kappa^{-1}(k_0),\ell'_{i+2})|\cdot|(m_{i+2},p_{i+2})|}{|(k_0,\ell'_{i+2})|\ \cdot\ |(m_{i+2},p_{i+2})|} \quad \text{(By Eq. (17))}\\
6.\quad &= \frac{|(\kappa^{-1}(k_0),\ell'_{i+2})|}{|(k_0,\ell'_{i+2})|}.
\end{aligned}$$

In Step 2, the ℓ_{i+1} (which came from Eq. (14)) is ℓ'_i with its orbits shifted and coordinates from n_0 and q_0 inserted. The coordinates ℓ'_i that are "shifted off" appear as coordinates of n_{i+1} and q_{i+1} and are "flushed out" in the cancellation in Step 3. In Step 5, the ℓ'_{i+2} comes from Eq. (16), so it is ℓ_{i+1} with its orbits shifted and factors of m_0 and p_0 inserted. The coordinates of ℓ_{i+1} that are "shifted off" appear as coordinates of m_{i+2} and p_{i+2}, and are flushed out in the cancellation in Step 6.

This procedure is iterated by inserting the quotient in Line 6 into the initial position in Line 1 and repeating. Beginning with $\frac{|(k_0,\ell'_0)|}{|(\kappa(k_0),\ell'_0)|}$ and iterating s times gives

$$\frac{|(k_0,\ell'_0)|}{|(\kappa(k_0),\ell'_0)|} = \frac{|(\kappa^{-s}(k_0),\ell'_{2s})|}{|(\kappa^{-s+1}(k_0),\ell'_{2s})|}.$$

If s is greater than the length of the longest orbit, then ℓ'_{2n} no longer has any of the original coordinates from ℓ'_0; they have all been flushed out and replaced by the fixed coordinates in n_0, m_0, p_0 and q_0. This means that $\frac{|(k_0,\ell'_0)|}{|(\kappa(k_0),\ell'_0)|}$ does not depend on ℓ'_0, so it is a constant C. (Recall k_0 is fixed in our discussion.)

Using this, Eq. (13) becomes

$$\frac{|(m,p)|}{|(m,q)|} = C\frac{|(n,p)|}{|(n,q)|}, \tag{18}$$

which holds for any m, n, p and q. Notice that the expression on the left depends only on p and q, because the variable m does not appear on the right. Thus there is a function Υ for which

$$\frac{|(m,p)|}{|(m,q)|} = \Upsilon(p,q).$$

Fixing vertices p_0, q_0, we use the above to get

$$|(m,p)| = |(m,q_0)|\cdot\Upsilon(p,q_0) \tag{19}$$

$$\begin{aligned}
|(m,q)| &= |(m,p_0)|\cdot\frac{1}{\Upsilon(p_0,q)}\\
&= |(m,q_0)|\cdot\frac{\Upsilon(p_0,q_0)}{\Upsilon(p_0,q)}.
\end{aligned} \tag{20}$$

Eq. (19) implies that for any m and p, the denominator of the (fully reduced) rational number $\Upsilon(p, q_0)$ divides $|(m, q_0)|$. Eq. (20) implies that for any m and q, the denominator of the (fully reduced) rational number $\frac{\Upsilon(p_0,q_0)}{\Upsilon(p_0,q)}$ divides $|(m, q_0)|$. If δ is the least common multiple of all such denominators, then δ divides $|(m, q_0)|$ for all m. By Eqs. (19) and (20),

$$\begin{aligned} |(m,p)| &= \frac{|(m,q_0)|}{\delta} \cdot \delta\Upsilon(p,q_0), \\ |(m,q)| &= \frac{|(m,q_0)|}{\delta} \cdot \delta\frac{\Upsilon(p_0,q_0)}{\Upsilon(p_0,q)}. \end{aligned}$$

Thus any $(m, x) \in V(B/R)$ has cardinality $|(m, x)| = \alpha(m) \cdot \beta(x)$ for the integer-valued functions

$$\begin{aligned} \alpha(m) &= \frac{|(m,q_0)|}{\delta} \\ \beta(x) &= \begin{cases} \delta\Upsilon(x,q_0) & \text{if } x \in V(P) \\ \delta\dfrac{\Upsilon(p_0,q_0)}{\Upsilon(p_0,x)} & \text{if } x \in V(Q). \end{cases} \end{aligned}$$

So indeed Proposition 2 yields $B = S' \times T'$ with $S'/R = S$ and $T'/R = T$.

As B is prime and T' is nontrivial (it has two partite sets), it follows that S' is a single vertex with a loop. Thus so is $S = S'/R$. Because $V(S) = V(M)$, it follows that $M = K_1$. A like argument shows $N = K_1$. Therefore we can eliminate these factors from Diagram (9) to obtain Diagram (21), below. (Note $L' \cong L$ because there are now no even orbits.)

$$\begin{array}{ccc} \overbrace{(K \,\square\, L\,)}^{S(A/R)} \,\square\, \overbrace{P}^{B_0} & \qquad\qquad & \overbrace{(K \,\square\, L\,)}^{S(A/R)} \,\square\, \overbrace{Q}^{B_1} \\ \Big\downarrow \quad \widetilde{\varphi}_0 & & \kappa\Big\downarrow \quad \widetilde{\varphi}_1 \\ \underbrace{(K \,\square\, L\,)}_{S(A'/R)} \,\square\, \underbrace{P}_{B'_0} & & \underbrace{(K \,\square\, L\,)}_{S(A'/R)} \,\square\, \underbrace{Q}_{B'_1} \end{array} \tag{21}$$

Here Eqs. (10) and (11) become $\widetilde{\varphi}_0\big((k,\ell),p\big) = \big((k,\lambda_0(\ell,p)),\ \pi(\ell,p)\big)$ and $\widetilde{\varphi}_1\big((k,\ell),q\big) = \big((\kappa(k),\lambda_1(\ell,q)),\ \chi(\ell,q)\big)$. Despite this mixing of coordinates we will next see that an isomorphism $B/R \to B'R$ can be recovered.

Part 4. Now we finish the proof by showing $B \cong B'$. We will begin by constructing an isomorphism $B/R \to B'/R$. Then we will lift it to an isomorphism $B \to B'$.

To do this, we first forge some notation. From Diagram (21) we see that the vertices of both B_0 and B'_0 (which are partite sets of B/R and B'/R respectively) are now identified with $V(P)$. To avoid ambiguity, we distinguish a vertex with a prime if belongs to B'_0. For instance, p and p' will represent the same vertex in P, but we interpret $p \in V(B_0)$ and $p' \in V(B'_0)$. The prime is simply a bookkeeping device to indicating whether p is in B_0 or B'_0. Similarly, the partite sets $V(B_1)$ and $V(B'_1)$ of B/R and B'/R, respectively, are now both identified with $V(Q)$. A single vertex q of Q might be denoted either as q or q', but we interpret $q \in V(B_1)$ and $q' \in V(B'_1)$.

Now it is easy to define an isomorphism $\widetilde{\Theta} : B/R \to B'/R$. Simply define $\widetilde{\Theta}(x) = x'$. In other words, $\widetilde{\Theta}(p) = p'$ and $\widetilde{\Theta}(q) = q'$. This is clearly a bijection.

To see that it is an isomorphism, refer to Remark 1, above (in Part 2 of the proof), which says that there is an edge $(m_0, p)(m_1, q) \in E(B/R)$ if and only if there is an edge $(n_0, p)(n_1, q) \in E(B'/R)$. With the factors M and N gone (and using our new "prime" notation), Remark 1 asserts that $pq \in E(B/R)$ if and only if $p'q' \in E(B'/R)$, so $\widetilde{\Theta}$ is indeed an isomorphism.

Next we lift this to an isomorphism $\Theta : B \to B'$. As always, vertices p and q of B/R are R-equivalence classes of B, with cardinalities $|p|$ and $|q|$. And vertices p' and q' of B'/R are R-equivalence classes of B', with cardinalities $|p'|$ and $|q'|$. Proposition 1 says $\widetilde{\Theta} : B/R \to B'/R$ lifts to an isomorphism $\Theta : B \to B$ provided that $|x| = |\widetilde{\Theta}(x)|$ for all $x \in V(B/R)$. By definition of $\widetilde{\Theta}$, this means we must show $|p| = |p'|$ and $|q| = |q'|$. We do this now.

Updating Eq. (18) by eliminating the factors that are no longer present, we get

$$\frac{|p|}{|q|} = C\frac{|p'|}{|q'|}, \quad \text{from which} \quad \frac{|p|}{|p'|} = C\frac{|q|}{|q'|}.$$

But the variable p on the left side of the later equation does not occur on the right, so this must be a constant D. That is,

$$\frac{|p|}{|p'|} = C\frac{|q|}{|q'|} = D.$$

Then for all p, q

$$|p| = D\,|p'| \quad \text{and} \quad |q| = \frac{D}{C}\,|q'|. \tag{22}$$

Denote the partite sets of B as X_0 and X_1; those of B' as X_0' and Y_0'. Arrange indexing so that $p \subseteq X_0$, $q \subseteq X_1$, $p' \subseteq X_0'$ and $q' \subseteq X_1'$. Eqs. (22) imply $|X_0| = D \cdot |X_0'|$ and $|X_1| = \frac{D}{C} \cdot |X_1'|$. As $\varphi : A \times B \to A' \times B'$ carries the partite set $V(A) \times X_0$ bijectively to the partite set $V(A') \times X_0'$, and because it carries partite set $V(A) \times X_1$ bijectively to partite set $V(A') \times X_1'$, we have

$$\begin{aligned} |V(A)| \cdot |X_0| &= |V(A')| \cdot |X_0'| = |V(A')| \cdot D \cdot |X_0|, \\ |V(A)| \cdot |X_1| &= |V(A')| \cdot |X_1'| = |V(A')| \cdot \frac{D}{C} \cdot |X_1|. \end{aligned}$$

From this, we get $D = D/C$, so $C = 1$ and Eq. (22) become

$$|p| = D \cdot |p'| \quad \text{and} \quad |q| = D \cdot |q'|. \tag{23}$$

We now show $D = 1$. Corollary 2 says the greatest common divisor of the multiset of numbers $|p|, |q|$ (where $p \in V(P)$ and $q \in V(Q)$) is 1. From this and Eq. (23) it follows that D is an integer. But then Eqs. (23) show that D divides the size of every R-equivalence class p or q of the prime graph B. Corollary 2 demands $D = 1$. Eqs. (23) become $|p| = |p'|$ and $|q| = |q'|$.

Proposition 1 now implies that the isomorphism $\widetilde{\theta} : B/R \to B'/R$ where $x \mapsto x'$ lifts to an isomorphism $\theta : B \to B'$. The theorem is proved. ■

References

[1] C.C. Chang, Cardinal factorization of finite relational structures, Fund. Math. 60 (1967) 251–269.
[2] R. Hammack, On unique prime bipartite factors of graphs, Discrete Math. 313 (9) (2013) 1018–1027.
[3] R. Hammack, W. Imrich, On Cartesian skeletons of graphs, Ars Math. Contemp. 2 (2009) 191–205.
[4] R. Hammack, W. Imrich, S. Klavžar (Eds.), Handbook of Product Graphs, second ed., CRC Press, Boca Raton, FLA, 2011.
[5] W. Imrich, Factoring cardinal product graphs in polynomial time, Discrete Math. 192 (1998) 119–144.
[6] R. McKenzie, Cardinal multiplication of structures with a reflexive relation, Fund. Math. 70 (1971) 59–101.
[7] G. Sabidussi, Graph multiplication, Math. Z. 72 (1960) 446–457.
[8] G.V. Vizing, The Cartesian product of graphs (Russian), Vyčisl Sistemy 9 (1963) 30–43.