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Cancellation of digraphs over the direct product

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ABSTRACT

In 1971 Lovász proved the following cancellation law concerning the direct product of digraphs. If A, B and C are digraphs, and C admits no homomorphism into a disjoint union of directed cycles, then $A \times C \cong B \times C$ implies $A \cong B$. On the other hand, if such a homomorphism exists, then there are pairs $A \ncong B$ for which $A \times C \cong B \times C$. This gives exact conditions on C that govern whether cancellation is guaranteed to hold or fail.

Left unresolved was the question of what conditions on A (or B) force $A \times C \cong B \times C \implies A \cong B$, or, more generally, what relationships between A and C (or B and C) guarantee this. Even if C has a homomorphism into a collection of directed cycles, can there still be restrictions on A and C that guarantee cancellation? We characterize the exact conditions.

We use a construction called the *factorial* $A!$ of a digraph A. Given digraphs A and C, the digraph $A!$ carries information that determines the complete set of solutions X to the digraph equation $A \times C \cong X \times C$. We state the exact conditions under which there is only one solution X (namely $X \cong A$) and that is the situation in which cancellation holds.

1. Introduction

The article [1] solves the following variation of the cancellation problem for the direct product of graphs: given graphs A and C, find all graphs B for which $A \times C \cong B \times C$. This can be regarded as a generalized cancellation law, for if there is only one such B, then $A \cong B$, that is, cancellation holds.

The analogous problem in the category of digraphs presents some special challenges, but the current article gives a complete solution. (Certain special cases were solved in [3,4].) Given arbitrary digraphs A and C, we describe all digraphs B for which $A \times C \cong B \times C$. In other words, we compute

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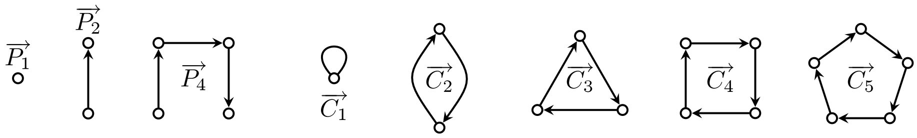

Fig. 1. Some digraphs.

all solutions X to the digraph equation $A \times C \cong X \times C$. If there is only one solution, then it can only be $X = A$. Thus, given an expression $A \times C \cong B \times C$, we can determine whether or not it follows necessarily that $A \cong B$ (i.e. whether cancellation holds).

We first fix the notation by recalling some relevant concepts. A *digraph* A is a binary relation $E(A)$ on a finite vertex set $V(A)$, that is, a subset $E(A) \subseteq V(A) \times V(A)$. We denote an ordered pair in $E(A)$ as $[x, y]$ and visualize it as an arrow pointing from x to y. Elements of $E(A)$ are called *arcs*. A reflexive arc $[x, x]$ is called a *loop*. A *graph* is a digraph that is symmetric (as a relation). We use the usual notation for graphs; in particular K_n is the complete graph on n vertices. By K_n^* we mean K_n with loops added to all of its vertices.

For a positive integer n, the *directed cycle* $\overrightarrow{C_n}$ is the digraph with vertices $\{0, 1, 2, \dots, n-1\}$ and arcs $[i, i+1]$ (arithmetic modulo n). Thus $\overrightarrow{C_1}$ consists of a single vertex with a loop, and $\overrightarrow{C_2} = K_2$. The *directed path* $\overrightarrow{P_n}$ is $\overrightarrow{C_n}$ with the arc $[n-1, 0]$ removed. See Fig. 1.

We denote the condition of X being a sub-digraph of A as $X \subseteq A$. A digraph A is *strongly connected* if for every pair x, y of its vertices there is a sub-digraph $\overrightarrow{P_n} \subseteq A$ beginning at x and ending at y. A digraph is *connected* if any x and y are joined by a path, each arc of which has arbitrary orientation. The *connected components* (respectively *strongly connected components*) of A are the maximal sub-digraphs of A that are connected (respectively strongly connected).

If A and B are digraphs, $A + B$ denotes their disjoint union. The disjoint union of n copies of A is denoted as nA. A *homomorphism* $\varphi : A \to B$ is a map $\varphi : V(A) \to V(B)$ for which $[x, y] \in E(A)$ implies $[\varphi(x), \varphi(y)] \in E(B)$. Two digraphs A, B are *homomorphically equivalent* if there are homomorphisms $A \to B$ and $B \to A$. An *isomorphism* is a bijective homomorphism. By $A \cong B$ we mean that A and B are isomorphic.

The *direct product* of two digraphs A and B is the digraph $A \times B$ whose vertex set is the Cartesian product $V(A) \times V(B)$ and whose arcs are the pairs $[(x, y), (x', y')]$ with $[x, x'] \in E(A)$ and $[y, y'] \in E(B)$. We assume the reader is familiar with direct products and homomorphisms. For standard references see [2,5].

2. Cancellation laws

Lovász [6] defines a digraph C to be a *zero divisor* if there exist non-isomorphic digraphs A and B for which $A \times C \cong B \times C$. For example, Fig. 2 shows that $\overrightarrow{C_3}$ is a zero divisor: if $A = \overrightarrow{C_3}$ and $B = 3\overrightarrow{C_1}$, then $A \ncong B$, yet $A \times \overrightarrow{C_3} \cong B \times \overrightarrow{C_3}$. (Both products are isomorphic to three copies of $\overrightarrow{C_3}$.) Here is the main result concerning zero divisors.

Theorem 1 (*Lovász [6, Theorem 8]*)**.** *A digraph C is a zero divisor if and only if there is a homomorphism $\varphi : C \to \overrightarrow{C_{p_1}} + \overrightarrow{C_{p_2}} + \overrightarrow{C_{p_3}} + \cdots + \overrightarrow{C_{p_k}}$ for prime numbers $p_1, p_2, \dots, p_k$.*

Thus, in particular, $\overrightarrow{C_n}$ with $n > 1$ is a zero divisor. (Even if n is not prime, there is an $\frac{n}{p}$-fold homomorphic cover $\varphi : \overrightarrow{C_n} \to \overrightarrow{C_p}$ for any prime divisor p of n.) Also each $\overrightarrow{P_n}$ is a zero divisor, for clearly there is a homomorphism $\overrightarrow{P_n} \to \overrightarrow{C_p}$ for any n and p.

Theorem 1 can be regarded as a cancellation law for the direct product, as it gives exact conditions on C under which $A \times C \cong B \times C$ necessarily implies $A \cong B$. However, it does not give a complete solution to the cancellation problem. We might also ask for conditions on A (or relationships between A and C) that force cancellation. For example, if $A = \overrightarrow{C_1}$ and C is nontrivial, then certainly $A \times C \cong B \times C$

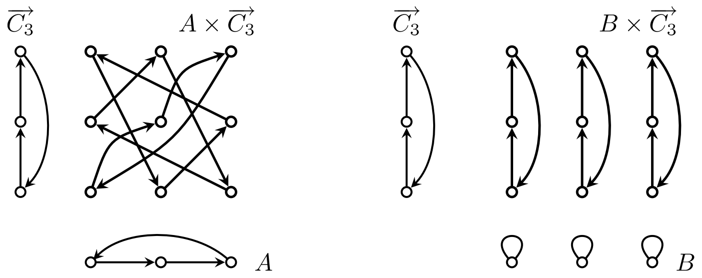

Fig. 2. Example of a zero divisor.

implies $A \cong B$ regardless of whether C is a zero divisor. What other A's have this property? This paper gives a complete solution.

We will need the following dichotomy involving zero divisors: Theorem 1 characterizes zero divisors as those digraphs C that admit a homomorphism $C \to \overrightarrow{C_{p_1}} + \overrightarrow{C_{p_2}} + \cdots + \overrightarrow{C_{p_k}}$. For each i there is a homomorphism $\overrightarrow{C_{p_i}} \to \overrightarrow{C_m}$, where $m = \gcd(p_1, \ldots, p_k)$. (Note $m = 1$ unless all the p_i are equal or $k = 1$.) Thus every zero divisor admits a homomorphism into some directed cycle. There may be only finitely many m for which homomorphisms $C \to \overrightarrow{C_m}$ exist. But for some C it may happen that there is a homomorphism $C \to \overrightarrow{C_m}$ for each positive integer m. Then, by taking $n > |V(C)|$, we see that C admits a homomorphism $C \to \overrightarrow{P_n} \subseteq \overrightarrow{C_n}$. Conversely, since there are homomorphisms $\overrightarrow{P_n} \to \overrightarrow{C_m}$ for any n and m, the existence of a homomorphism $C \to \overrightarrow{P_n}$ guarantees a homomorphism $C \to \overrightarrow{C_m}$ for every m. Therefore zero divisors C can be divided into two distinct and mutually exclusive types: on the one hand there are those that admit a homomorphism $C \to \overrightarrow{P_n}$ for some n (and thus homomorphisms $C \to \overrightarrow{C_p}$ for infinitely many p); on the other hand there are those that admit homomorphisms $C \to \overrightarrow{C_m}$ for only finitely many m. These facts suggest the following definition.

Definition 1. A zero divisor is of *Type P* if there is a homomorphism $C \to \overrightarrow{P_m}$ for some m. If C is not of Type P, then it admits homomorphisms $C \to \overrightarrow{C_m}$ for only finitely many m, and we say C is of *Type C*.

Remark 1. If a zero divisor C is of Type P, then there is a smallest n for which there is a homomorphism $C \to \overrightarrow{P_n}$. If C is of Type C, there is a largest n for which $C \to \overrightarrow{C_n}$; if C is *connected*, Theorem 1 implies $n > 1$.

Zero divisors of Type P having homomorphisms into P_1 are spectacularly uninteresting, as they have no arcs. For them, the cancellation problem is trivial: $A \times C \cong B \times C$ if and only if $|V(A)| = |V(B)|$. We will have nothing further to say about this situation; henceforward we tacitly assume that any zero divisor has at least one arc.

Our methods will make frequent reference to the two types of zero divisors given by Definition 1. We will also require the following theorems due to Lovász. (See [5] for a very readable proof of Theorem 2 and related topics.)

Theorem 2 (*Lovász [6, Theorem 6]*)**.** *Let A, B, C and D be digraphs. If $A \times C \cong B \times C$ and there is a homomorphism $D \to C$, then $A \times D \cong B \times D$.*

Theorem 3 (*Lovász [6, Theorem 7]*)**.** *Let A, B, C be digraphs. If $A \times C \cong B \times C$, then there is an isomorphism from $A \times C$ to $B \times C$ of the form $(x, c) \mapsto (\varphi_c(x), c)$, where $\varphi_c : V(A) \to V(B)$ is a map that depends on c.*

3. Permuted digraphs

Given a digraph A, let $S_{V(A)}$ denote the symmetric group on $V(A)$. (That is, $S_{V(A)}$ is the set of bijections from $V(A)$ to itself.) The next definition is central to the remainder of this paper. For a permutation $\pi \in S_{V(A)}$, define the *permuted digraph* A^π as follows.

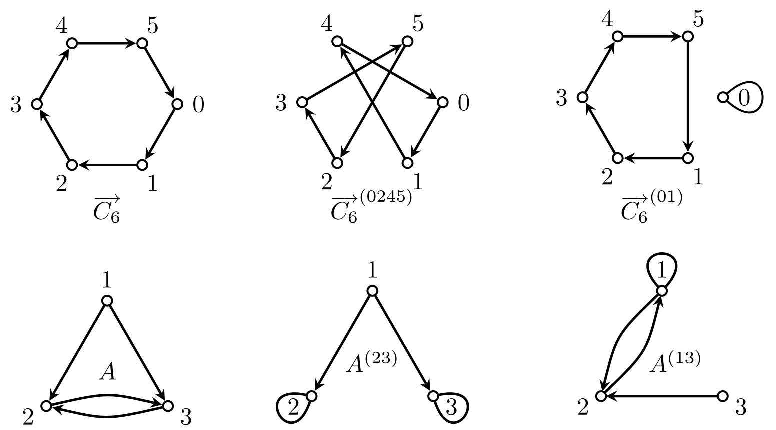

Fig. 3. Examples of permuted digraphs.

Definition 2. If A is a digraph and $\pi \in S_{V(A)}$, the *permuted digraph* A^{π} has vertices $V(A^{\pi}) = V(A)$ and arcs $E(A^{\pi}) = \{[x, \pi(y)] : [x, y] \in E(A)\}$. Thus $[x, y] \in E(A)$ if and only if $[x, \pi(y)] \in E(A^{\pi})$, and $[x, y] \in E(A^{\pi})$ if and only if $[x, \pi^{-1}(y)] \in E(A)$.

Fig. 3 shows several examples. In the upper part of the figure, the cyclic permutation (0245) of the vertices of $\overrightarrow{C_6}$ yields a permuted graph $\overrightarrow{C_6}^{(0245)} = 2\overrightarrow{C_3}$. The permuted digraph $\overrightarrow{C_6}^{(01)}$ is also shown. The lower part of the figure shows a digraph A and two of its permuted digraphs. For another example, note that $A^{\mathrm{id}} = A$ for any digraph A. We remark that it may be possible that $A^{\pi} \cong A$ for some non-identity permutation π. For instance, $\overrightarrow{C_6}^{(024)} \cong \overrightarrow{C_6}$.

The significance of permuted digraphs is given by the following proposition, which asserts that whenever $A \times C \cong B \times C$, it necessarily follows that B is a permuted digraph of A.

Proposition 1. *Let A, B and C be digraphs, where C has at least one arc. If $A \times C \cong B \times C$, then $B \cong A^{\pi}$ for some permutation $\pi \in S_{V(A)}$.*

Proof. Suppose $A \times C \cong B \times C$, and C has at least one arc. Because C has an arc, there is a homomorphism $\overrightarrow{P_2} \to C$, and Theorem 2 yields an isomorphism $\varphi : A \times \overrightarrow{P_2} \to B \times \overrightarrow{P_2}$. In turn, Theorem 3 guarantees that this isomorphism has the form $(x, \varepsilon) \mapsto (\varphi_{\varepsilon}(x), \varepsilon)$, where $\varepsilon \in \{0, 1\} = V(\overrightarrow{P_2})$, and each φ_{ϵ} is a map from $V(A)$ to $V(B)$. As φ is an isomorphism, it follows immediately that φ_0 and φ_1 are bijections. Hence $\varphi_0^{-1}\varphi_1 : V(A) \to V(A)$ is a permutation in $S_{V(A)}$. We now show that $\varphi_0 : V(A^{\varphi_0^{-1}\varphi_1}) \to V(B)$ is an isomorphism. Simply observe that

$$
\begin{array}{rll}
 & [x, y] \in E(A^{\varphi_0^{-1}\varphi_1}) & \\
\Longleftrightarrow & [x, (\varphi_0^{-1}\varphi_1)^{-1}(y)] \in E(A) & (\text{definition of } A^{\varphi_0^{-1}\varphi_1}) \\
\Longleftrightarrow & [x, \varphi_1^{-1}\varphi_0(y)] \in E(A) & \\
\Longleftrightarrow & [(x, 0), (\varphi_1^{-1}\varphi_0(y), 1)] \in E(A \times \overrightarrow{P_2}) & (\text{definition of } A \times \overrightarrow{P_2}) \\
\Longleftrightarrow & [(\varphi_0(x), 0), (\varphi_1\varphi_1^{-1}\varphi_0(y), 1)] \in E(B \times \overrightarrow{P_2}) & (\text{apply isomorphism } \varphi) \\
\Longleftrightarrow & [(\varphi_0(x), 0), (\varphi_0(y), 1)] \in E(B \times \overrightarrow{P_2}) & \\
\Longleftrightarrow & [\varphi_0(x), \varphi_0(y)] \in E(B) & (\text{definition of } B \times \overrightarrow{P_2}).
\end{array}
$$

Thus $A^{\varphi_0^{-1}\varphi_1} \cong B$, and the proof is complete. □

In general, the converse of Proposition 1 is (as we shall see) false. Depending on A and C, not every π yields a digraph $B = A^{\pi}$ for which $A \times C \cong B \times C$. In addition, it is possible that $\pi \neq \sigma$ but $A^{\pi} \cong A^{\sigma}$. Towards clarifying these issues, we next introduce a construction called the *factorial* of a digraph.

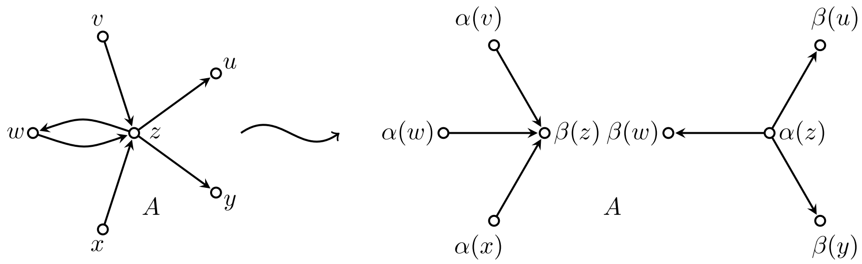

Fig. 4. Action of an arc $[\alpha, \beta] \in E(A!)$ on the neighborhood of a vertex $z \in V(A)$.

4. The digraph factorial

The key to our main results is the *digraph factorial*, an operation on digraphs that is somewhat analogous to the factorial operation on integers. It was introduced in [4], but the interpretation here is extended significantly. Recall that $S_{V(A)}$ is the set of permutations of the vertices of a digraph A.

Definition 3. Given a digraph A, its *factorial* is another digraph, denoted as $A!$, and defined as follows. The vertex set is $V(A!) = S_{V(A)}$. For the edge set, we define $[\alpha, \beta] \in E(A!)$ provided that $[x, y] \in E(A) \iff [\alpha(x), \beta(y)] \in E(A)$ for all pairs $x, y \in V(A)$.

Observe that the definition implies that there is a loop $[\alpha, \alpha]$ at $\alpha \in V(A!)$ if and only if α is an automorphism of A. In particular, any $A!$ has a loop at the identity id. We remark also that (at least for finite digraphs) Definition 3 can be weakened by replacing the "$\iff$" with a "$\Rightarrow$". It follows that $A!$ is the subgraph of the digraph exponential A^A induced on the bijections $A \to A$.

Our first example explains the origin of our term "factorial". Let K_n^* be the complete (symmetric) graph with a loop at each vertex, and note that

$$K_n^*! \cong K_{n!}^* \cong K_n^* \times K_{n-1}^* \times K_{n-2}^* \times \cdots \times K_3^* \times K_2^* \times K_1^*.$$

For less obvious computations, it is helpful to keep in mind the following interpretation of $E(A!)$. Any arc $[\alpha, \beta] \in E(A!)$ can be regarded as a permutation of the arcs of A, where $[\alpha, \beta]([x, y]) = [\alpha(x), \beta(z)]$. This permutation preserves in-incidences and out-incidences in the following sense: given two arcs $[x, y], [x, z]$ of A that have a common tail, $[\alpha, \beta]$ carries them to the two arcs $[\alpha(x), \beta(y)], [\alpha(x), \beta(z)]$ of A with a common tail. Given two arcs $[x, y], [z, y]$ with a common tip, $[\alpha, \beta]$ carries them to the two arcs $[\alpha(x), \beta(y)], [\alpha(z), \beta(y)]$ of A with a common tip.

Bear in mind, however, that even if the tip of $[x, y]$ meets the tail of $[y, z]$, then the arcs $[\alpha, \beta]([x, y])$ and $[\alpha, \beta]([y, z])$ need not meet; they can be quite far apart in A. To illustrate these ideas, Fig. 4 shows the effect of a typical $[\alpha, \beta]$ on the arcs incident with a typical vertex z of A.

We use these ideas in the next example, which will be used later. It also illustrates that the factorial can have just a single arc [id, id].

Example 1. Let T_n denote the (unique) transitive tournament on n vertices. This digraph has distinct out-degrees $n-1, n-2, \ldots, 0$ and distinct in-degrees $0, 1, 2, \ldots, n-1$. The above discussion implies that for a given $[\alpha, \beta] \in E(T_n!)$, the out-degree of any $x \in V(T_n)$ equals the out-degree of $\alpha(x)$. Hence $\alpha = \mathrm{id}$. The same argument involving in-degrees gives $\beta = \mathrm{id}$. Therefore $T_n!$ has $n!$ vertices but only one arc [id, id].

Fig. 5 shows $T_3!$, plus two other examples of factorials.

By Definition 3, $[\alpha, \beta] \in E(A!)$ if and only if $[\alpha^{-1}, \beta^{-1}] \in E(A!)$. In fact, it is immediate that $E(A!)$ is a group with identity [id, id] and multiplication $[\alpha, \beta][\gamma, \delta] = [\alpha\gamma, \beta\delta]$. We also have $[\alpha, \beta]^{-1} = [\alpha^{-1}, \beta^{-1}]$. Moreover, $\mathrm{Aut}(A)$ embeds as a subgroup of $E(A!)$, for it is the set of loops $[\alpha, \alpha]$ of $E(A!)$. In this sense, $E(A!)$ can be regarded as an extension of $\mathrm{Aut}(A)$. It carries all the information of $\mathrm{Aut}(A)$, plus more.

As an example, note that $E(K_n^*!)$ consists of all elements $[\alpha, \beta]$ where $\alpha, \beta \in S_n$, so we see that $E(K_n^*!)$ is isomorphic to the group product $S_n \times S_n$. At another extreme, $E(T_n!)$ is the trivial group. The reader may verify that $E(A!)$ in Fig. 5 is the Klein four-group. Referring again to Fig. 5, observe that $E(\overrightarrow{C_3}!)$ is the symmetric group S_3.

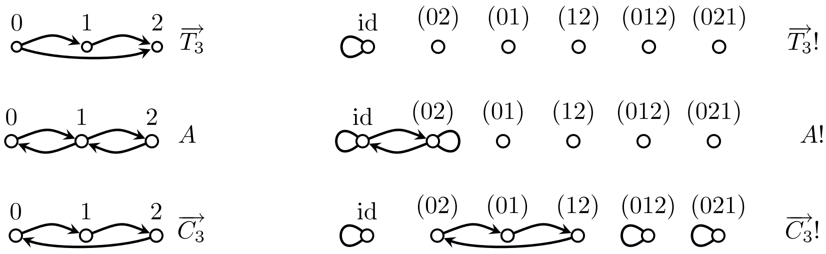

Fig. 5. Examples of digraphs (left) and their factorials (right).

5. The group action of $E(A!)$ on $V(A!)$

The group $E(A!)$ acts on $V(A!) = S_{V(A)}$ as $[\alpha, \beta]\cdot\pi = \alpha\pi\beta^{-1}$. This action determines the situations under which $A^\pi \cong A^\sigma$.

Proposition 2. *Suppose A is a digraph, and $\pi, \sigma \in V(A!)$. Then $A^\pi \cong A^\sigma$ if and only if π and σ are in the same $E(A!)$-orbit of $V(A!)$.*

Proof. Let $\varphi : A^\pi \to A^\sigma$ be an isomorphism. For any $x, y \in V(A)$, we have

$$\begin{array}{lll} & [x, y] \in E(A) & \\ \Longleftrightarrow & [x, \pi(y)] \in E(A^\pi) & \text{(definition of } A^\pi\text{)} \\ \Longleftrightarrow & [\varphi(x), \varphi\pi(y)] \in E(A^\sigma) & \text{(apply isomorphism } \varphi\text{)} \\ \Longleftrightarrow & [\varphi(x), \sigma^{-1}\varphi\pi(y)] \in E(A) & \text{(definition of } A^\sigma\text{).} \end{array}$$

From this, and the definition of $A!$, it follows that $[\varphi,\ \sigma^{-1}\varphi\pi] \in E(A!)$. We then have

$$[\varphi,\ \sigma^{-1}\varphi\pi]\cdot\pi = \varphi\pi\pi^{-1}\varphi^{-1}\sigma = \sigma,$$

so π and σ are indeed in the same orbit.

Conversely, suppose π and σ are in the same $E(A!)$-orbit of $V(A!)$, so $\sigma = \alpha\pi\beta^{-1}$ for some $[\alpha, \beta] \in E(A!)$. We claim that $\alpha : A^\pi \to A^{\alpha\pi\beta^{-1}}$ is an isomorphism. Indeed,

$$\begin{array}{lll} & [x, y] \in E(A^\pi) & \\ \Longleftrightarrow & [x, \pi^{-1}(y)] \in E(A) & \text{(definition of } A^\pi\text{)} \\ \Longleftrightarrow & [\alpha(x), \beta\pi^{-1}(y)] \in E(A) & ([\alpha, \beta] \in E(A!)) \\ \Longleftrightarrow & [\alpha(x), \alpha\pi\beta^{-1}\beta\pi^{-1}(y)] \in E(A^{\alpha\pi\beta^{-1}}) & \text{(definition of } A^{\alpha\pi\beta^{-1}}\text{)} \\ \Longleftrightarrow & [\alpha(x), \alpha(y)] \in E(A^{\alpha\pi\beta^{-1}}), & \end{array}$$

and the assertion follows. □

Given an arc $[\alpha, \beta] \in E(A!)$, we have $[\alpha, \beta]\cdot\beta = \alpha$. The previous proposition then assures $A^\alpha \cong A^\beta$, and therefore yields the following corollaries.

Corollary 1. *If two elements $\pi, \sigma \in V(A!)$ are in the same component of $A!$, then $A^\pi \cong A^\sigma$.*

As $A = A^{\text{id}}$, Corollary 1 combines with Proposition 1 to yield the following sufficient condition for cancellation. (Exact conditions are more subtle, but we will lay them out in the next two sections.)

Corollary 2. *If $A!$ is connected, then $A \times C \cong B \times C$ implies $A \cong B$ (whether or not C is a zero divisor).*

6. Main results: zero divisors of Type P

Recall that Definition 1 divides zero divisors into Types P and C. We now investigate zero divisors of Type P, that is, those that admit a homomorphism $C \to \overrightarrow{P}_n$. (Zero divisors of Type C are addressed in the subsequent section.)

The next theorem characterizes, given A and a zero divisor C of Type P, all digraphs B for which $A \times C \cong B \times C$. List the vertices of $\overrightarrow{P_n}$ (consecutively) as $0, 1, \ldots, n-1$. A *directed walk of length* n in a digraph is a sequence of n arcs $[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$ (not necessarily distinct).

Theorem 4. *Suppose A, B and C are digraphs, and C is a zero divisor (with at least one arc) of Type P. Let $n \geq 2$ be the smallest integer for which there is a homomorphism $\rho : C \to \overrightarrow{P_n}$. Then $A \times C \cong B \times C$ if and only if $B \cong A^{\pi}$, where π is a vertex on a directed walk of length $n-2$ in the factorial $A!$.*

Proof. Suppose that $B \cong A^{\pi}$, where π is a vertex on a directed walk of length $n-2$ in $A!$. List the walk's vertices consecutively as $\pi_1, \pi_2, \ldots, \pi_{n-1}$, where $\pi = \pi_i$ for some i. By Corollary 1, $B \cong A^{\pi_1}$, so we just need to show $A \times C \cong A^{\pi_1} \times C$. Define a map $\varphi : V(A \times C) \to V(A^{\pi_1} \times C)$ as

$$\varphi(x, c) = \begin{cases} \big(\pi_1\pi_2 \cdots \pi_{\rho(c)}(x), c\big) & \text{if } 1 \leq \rho(c) \leq n-1, \\ (x,\ c) & \text{if } \rho(c) = 0. \end{cases}$$

Clearly this is a bijection, because each π_i is a permutation of $V(A) = V(A^{\pi_1})$. We need to show that it is an isomorphism, that is, we must show $[(x, c), (y, c')] \in E(A \times C) \iff [\varphi(x, c), \varphi(y, c')] \in E(A^{\pi_1} \times C)$.

If either $[(x, c), (y, c')] \in E(A \times C)$ or $[\varphi(x, c), \varphi(y, c')] \in E(A^{\pi_1} \times C)$, then $cc' \in E(C)$ and the definition of ρ then implies $\rho(c') = \rho(c) + 1$. With this in mind we consider two cases.

Case I. Suppose $[(x, c), (y, c')]$ satisfies $\rho(c) \neq 0$. Consider the following product of arcs in $E(A!)$:

$$[\pi_1, \pi_2][\pi_2, \pi_3][\pi_3, \pi_4] \cdots [\pi_{\rho(c)}, \pi_{\rho(c')}] = [\pi_1\pi_2 \cdots \pi_{\rho(c)},\ \pi_2\pi_3 \cdots \pi_{\rho(c')}].$$

This product is itself an arc in $E(A!)$. Therefore we have

$$\begin{aligned} [x, y] \in E(A) &\iff [\pi_1\pi_2 \cdots \pi_{\rho(c)}(x),\ \pi_2\pi_3 \cdots \pi_{\rho(c')}(y)] \in E(A) \\ &\iff [\pi_1\pi_2 \cdots \pi_{\rho(c)}(x),\ \pi_1\pi_2\pi_3 \cdots \pi_{\rho(c')}(y)] \in E(A^{\pi_1}). \end{aligned} \tag{1}$$

From this it follows that

$$\begin{aligned} & [(x, c), (y, c')] \in E(A \times C) \\ \iff & [(\pi_1\pi_2 \cdots \pi_{\rho(c)}(x),\ c),\ (\pi_1\pi_2\pi_3 \cdots \pi_{\rho(c')}(y), c')] \in E(A^{\pi_1} \times C) \\ \iff & [\varphi(x, c),\ \varphi(y, c)] \in E(A^{\pi_1} \times C). \end{aligned}$$

Case II. Suppose $[(x, c), (y, c')]$ satisfies $\rho(c) = 0$, so $\rho(c') = 1$. Note that $[(x, c), (y, c')] \in E(A \times C)$ if and only if $[(x, c), (\pi_1(y), c')] \in E(A^{\pi_1} \times C)$, which (by definition of φ) is the same as $[\varphi(x, c), \varphi(y, c')] \in E(A^{\pi_1} \times C)$.

We have now established an isomorphism $\varphi : A \times C \to A^{\pi_1} \times C$.

Conversely, suppose $A \times C \cong B \times C$. We must show that $B \cong A^{\pi}$, where π is a vertex of a directed walk of length $n-2$ in $A!$.

Let $\varphi : A \times C \to B \times C$ be an isomorphism. By Theorem 3, we can (and do) assume that φ has form $\varphi(x, c) = (\varphi_c(x), c)$, where $\varphi_c : V(A) \to V(B)$ is a bijection for each c. Thus, for an arbitrary fixed arc $[c, c'] \in E(C)$ we have $[(x, c), (y, c')] \in E(A \times C)$ if and only if $[(\varphi_c(x), c), (\varphi_{c'}(y), c')] \in E(B \times C)$. It follows that for any $[c, c'] \in E(C)$ we have

$$[x, y] \in E(A) \iff [\varphi_c(x), \varphi_{c'}(y)] \in E(B), \tag{2}$$

and from this we get

$$[x, y] \in E(B) \iff [\varphi_c^{-1}(x), \varphi_{c'}^{-1}(y)] \in E(A). \tag{3}$$

As n is the smallest integer for which there is a homomorphism $C \to \overrightarrow{P_n}$, it readily follows that some component C' of C satisfies $\rho(V(C')) = V(\overrightarrow{P_n})$. Thus C has a path P (not necessarily directed) on consecutive vertices $c_0, c_1, c_2, \ldots, c_k$ for which $\rho(c_0) = 0$ and $\rho(c_k) = n-1$. See Fig. 6.

Now we are going to construct a directed walk of length $n-2$ in $A!$. We begin with a certain labeling of the endpoints of the arcs in P with maps φ_c from the definition of φ. For a given $i \in V(\overrightarrow{P_n})$, suppose

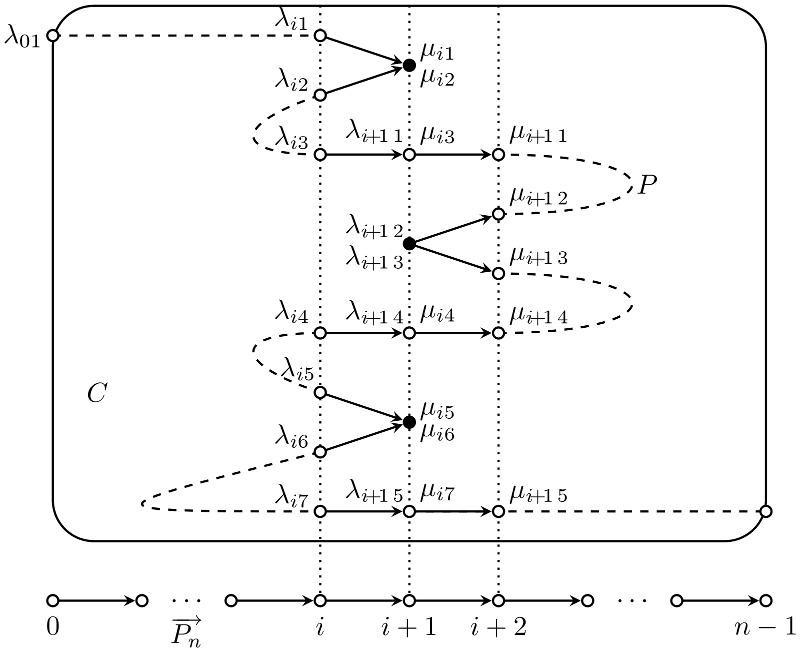

Fig. 6. Arcs of P that ρ sends to $[i, i+1]$ and $[i+1, i+2]$.

$[c, c'] \in E(P)$ is the jth arc of P for which $\rho([c, c']) = [i, i+1]$. Label c with the map $\lambda_{ij} = \varphi_c$, and label c' with the map $\mu_{ij} = \varphi_{c'}$. Fig. 6 illustrates this. Notice that the sinks of P get two labels (both μ's), as do the sources (both λ's). The first vertex of P gets the single label $\lambda_{01} = \varphi_{c_0}$, and the last vertex c_k is labeled with $\mu_{(n-2)s} = \varphi_{c_{k-1}}$, for some $1 \leq s \leq k$. Any other vertex c_t of P gets two labels, a $\lambda_{\rho(c_t)s}$ and a $\mu_{\rho(c_t)-1,s}$. Any two labels of the same vertex are equal functions. (This labeling is inspired by a similar one in [6], used in a different context.)

Now, for any i, there is an odd number of labels $\lambda_{i1}, \lambda_{i2}, \lambda_{i3}, \ldots, \lambda_{i\ell_i}$, and the same (odd) number of labels $\mu_{i1}, \mu_{i2}, \mu_{i3}, \ldots, \mu_{i\ell_i}$. For each $0 \leq i \leq n-2$, define the maps

$$L_i = \lambda_{i1}\lambda_{i2}^{-1}\lambda_{i3}\lambda_{i4}^{-1} \cdots \lambda_{i\ell_i} \tag{4}$$

$$M_i = \mu_{i1}\mu_{i2}^{-1}\mu_{i3}\mu_{i4}^{-1} \cdots \mu_{i\ell_i}. \tag{5}$$

Recall that, for any $1 \leq j \leq \ell_i$, the terms λ_{ij} and μ_{ij} that appear in these expressions are bijections $\lambda_{ij} = \varphi_c$ and $\mu_{ij} = \varphi_{c'}$ for some arc $[c, c'] \in E(P) \subseteq E(C)$. By applying Equivalences (2) and (3) successively (and an odd number of times), we get

$$[x, y] \in E(A) \iff [L_i(x), M_i(y)] \in E(B). \tag{6}$$

We now claim that $M_i = L_{i+1}$: in Eq. (4), any pair of consecutive λ's that correspond to a source in P cancel. Likewise, in Eq. (5), any pair of consecutive μ's that correspond to a sink in P cancel. Once these pairs have been removed, the remaining terms in the expressions for L_i and M_i match. (Heuristically, we can think of the black vertices in Fig. 6 as being eliminated.) For example, in Fig. 6, we have

$$L_{i+1} = \lambda_{i+1\,1}\,(\lambda_{i+1\,2}^{-1}\,\lambda_{i+1\,3})\,\lambda_{i+1\,4}^{-1}\,\lambda_{i+1\,5} = \lambda_{i+1\,1}\,\lambda_{i+1\,4}^{-1}\,\lambda_{i+1\,5}$$

$$M_i = (\mu_{i1}\,\mu_{i2}^{-1})\,\mu_{i3}\,\mu_{i4}^{-1}\,(\mu_{i5}\,\mu_{i6}^{-1})\,\mu_{i7} = \mu_{i3}\quad\mu_{i4}^{-1}\quad\mu_{i7}.$$

Now, $\lambda_{i+1\,1} = \mu_{i3}$ because they label the same vertex. Likewise, $\lambda_{i+1\,4} = \mu_{i4}$ and $\lambda_{i+1\,5} = \mu_{i7}$. Then $M_i = L_{i+1}$, as claimed.

Equivalence (6) can now be updated as

$$[x, y] \in E(A) \iff [L_i(x), L_{i+1}(y)] \in E(B), \tag{7}$$

where $0 \le i \le n-3$. In fact, the indexing allows us to define $L_{n-1} = M_{n-2}$, so Equivalence (7) actually holds for $0 \le i \le n-1$. From (7) we get

$$[x, y] \in E(A) \iff [L_{i-1}^{-1}L_i(x), L_i^{-1}L_{i+1}(y)] \in E(A), \tag{8}$$

for each $1 \le i \le n-2$. Therefore we have the following walk of length $n-2$ in $A!$, whose first vertex is $L_0^{-1}L_1$.

$$[L_0^{-1}L_1,\ L_1^{-1}L_2], [L_1^{-1}L_2,\ L_2^{-1}L_3], [L_2^{-1}L_3,\ L_3^{-1}L_4], \ldots, [L_{n-3}^{-1}L_{n-2},\ L_{n-2}^{-1}L_{n-1}].$$

To finish the proof, we show that $L_0 : A^{L_0^{-1}L_1} \to B$ is an isomorphism. Equivalence (7) yields $[x, y] \in E(A) \iff [L_0(x), L_1(y)] \in E(B)$. Using this, $[x, y] \in E(A^{L_0^{-1}L_1})$ if and only if $[x, L_1^{-1}L_0(y)] \in E(A)$, if and only if $[L_0(x), L_1L_1^{-1}L_0(y)] \in E(B)$, if and only if $[L_0(x), L_0(y)] \in E(B)$. $\square$

Given a digraph A and a zero divisor C that admits $C \to \overrightarrow{P_n}$, Theorem 4 describes a complete collection of digraphs B for which $A \times C \cong B \times C$. Of course it is possible that some (possibly all) of these B are isomorphic. We next describe a means of constructing the exact set of isomorphism classes of such B. Combining the previous theorem with Proposition 2 yields the following.

Corollary 3. *Let A and C be digraphs, and C be a zero divisor of Type P, and $n \ge 2$ be the least integer for which there is a homomorphism $C \to \overrightarrow{P_n}$. Then the distinct (up to isomorphism) digraphs B for which $A \times C \cong B \times C$ are obtained as follows: let Υ_{n-2} be the set of vertices of $A!$ that lie on a directed walk of length $n-2$. Take a maximal set of elements $\pi_1, \pi_2, \ldots, \pi_k \in \Upsilon_{n-2}$ that are in distinct orbits of the $E(A!)$-action on $V(A!)$. Then the digraphs B for which $A \times C \cong B \times C$ are precisely $B \cong A^{\pi_1}, A^{\pi_2}, \ldots, A^{\pi_k}$.*

Cancellation holds (that is, $A \times C \cong B \times C$ guarantees $A \cong B$) if and only if $k = 1$.

By Theorem 4, if C admits a homomorphism into $\overrightarrow{P_2}$ (which, given that C has at least one arc, implies it is homomorphically equivalent to $\overrightarrow{P_2}$), then $A \times C \cong B \times C$ if and only if $B \cong A^\pi$, where π is a vertex of $A!$ on a walk of length 0. In this case there are no restrictions whatsoever on π; it can be any vertex of $A!$. Thus there can be potentially $|V(A)|!$ different $B \cong A^\pi$. We summarize this in a corollary, proved as a free-standing result in [4].

Corollary 4. *If C is homomorphically equivalent to $\overrightarrow{P_2}$, then $A \times C \cong B \times C$ if and only if $B \cong A^\pi$ for some permutation π of $V(A)$.*

Here is an application of the previous two corollaries that illustrates an extreme failure of cancellation. Let T_n be the transitive tournament on n vertices. Example 1 in Section 4 showed that $T_n!$ has $n!$ vertices and a single arc [id, id]. Therefore each $E(A!)$-orbit of $V(A!)$ consists of a single permutation. Also $\Upsilon_0 = V(A!)$. Thus, if C is a zero divisor that admits a homomorphism into P_2, then there are exactly $n!$ distinct digraphs T_n^π for which $T_n \times C \cong T_n^\pi \times C$. By Proposition 1, this is the maximum number possible. (But merely replace C with a zero divisor that admits a homomorphism into P_n with $n > 2$; then $\Upsilon_{n-2} = \{\text{id}\}$ and cancellation holds!)

7. Main results: zero divisors of Type C

The previous section treated all zero divisors of Type P. We now develop a parallel theory for those of Type C. Our reasoning follows that of the previous section, except that the situation here is somewhat richer. We will need the following definition.

A *null-walk* in $A!$ is a directed closed walk $[\pi_0, \pi_1], [\pi_1, \pi_2], [\pi_2, \pi_3], \ldots, [\pi_{n-1}, \pi_0]$ for which $[\pi_0, \pi_1][\pi_1, \pi_2][\pi_2, \pi_3] \cdots [\pi_{n-1}, \pi_0] = [\text{id}, \text{id}]$. Null-walks will play a role analogous to that of the directed walk of length $n-2$ in the previous section. Although the conditions of the definition may seem restrictive, null-walks are not particularly rare. Take any directed closed walk in $A!$ multiply its vertices consecutively to get a permutation σ, and traverse the walk $|\sigma|$ times; the result is a null-walk.

Our first result is analogous to one direction of Theorem 4.

Proposition 3. *Suppose a digraph C admits a homomorphism $\rho : C \to \overrightarrow{C_n}$. If $\pi \in V(A!)$ is on a null-walk of length n, then $A \times C \cong A^{\pi} \times C$.*

Proof. Let C, ρ and π be as stated, and say π is on the null-walk

$$[\pi_0, \pi_1], [\pi_1, \pi_2], [\pi_2, \pi_3], \dots, [\pi_{n-1}, \pi_0].$$

By Corollary 1, $A^{\pi} \cong A^{\pi_0}$, so it suffices to prove $A \times C \cong A^{\pi_0} \times C$. Define a map $\varphi : V(A \times C) \to V(A^{\pi_0} \times C)$ as

$$\varphi(x, c) = \big(\pi_0\pi_1\pi_2 \cdots \pi_{\rho(c)}(x),\ c\big).$$

Clearly this is a bijection, because each π_i is a permutation of $V(A) = V(A^{\pi_0})$. We need to show that it is an isomorphism. Note that the product

$$[\pi_0, \pi_1][\pi_1, \pi_2][\pi_2, \pi_3] \cdots [\pi_{\rho(c)}, \pi_{\rho(c)+1}] = [\pi_0\pi_1 \cdots \pi_{\rho(c)},\ \pi_1\pi_2 \cdots \pi_{\rho(c)+1}]$$

is an arc in $A!$. Using this and the fact that $\rho(c') = \rho(c) + 1$ when $[c, c'] \in E(C)$, it follows that

$$\begin{aligned}
& [(x, c), (y, c')] \in E(A \times C) \\
\iff\ & [(\pi_0\pi_1 \cdots \pi_{\rho(c)}(x), c), (\pi_1\pi_2 \cdots \pi_{\rho(c)+1}(y), c')] \in E(A \times C) \\
\iff\ & [(\pi_0\pi_1 \cdots \pi_{\rho(c)}(x), c), (\pi_0\pi_1\pi_2 \cdots \pi_{\rho(c)+1}(y), c')] \in E(A^{\pi_0} \times C) \\
\iff\ & [(\pi_0\pi_1 \cdots \pi_{\rho(c)}(x), c), (\pi_0\pi_1\pi_2 \cdots \pi_{\rho(c')}(y), c')] \in E(A^{\pi_0} \times C) \\
\iff\ & [\varphi(x, c),\ \varphi(y, c')] \in E(A^{\pi_0} \times C).
\end{aligned}$$

(The last step required the null-walk hypothesis. If $\rho(c) = n - 1$, then $\rho(c') = 0$, and we need $\pi_0\pi_1\pi_2 \cdots \pi_{n-1} = \mathrm{id}$ so that $(\pi_0\pi_1\pi_2 \cdots \pi_{\rho(c')}(y), c')$ reduces to $(\pi_0(y), c') = \varphi(y, c')$.) □

Developing an analog of the *converse* direction of Theorem 4 requires a lemma. Recall that the proof of that theorem involved a path P in C with an odd number of arcs that ρ sends to $[i, i+1]$. The next lemma will provide analogous conditions for our current setting.

Lemma 5. *Suppose C is a digraph and $n \geq 2$ is the largest integer for which there is a homomorphism $\rho : C \to \overrightarrow{C_n}$. Then there is a closed walk W (not necessarily directed) in C that has, for each i, an odd number of arcs $[c, c']$ with $[\rho(c), \rho(c')] = [i, i+1]$.*

Proof. Define an integer-valued function f on the walks of C as follows. Suppose that in traversing a walk W we cross arcs k times in the proper (tail-to-tip) orientation, and ℓ times in the reverse (tip-to-tail) orientation. Then $f(W) = k - \ell$.

If the last vertex of W is the first vertex of a walk X, we denote their concatenation as $W + X$; then $f(W+X) = f(W) + f(X)$. Also, let $-W$ denote the walk W traversed in the opposite direction; then $f(-W) = -f(W)$. If W and X have the same terminal vertex, then $W - X$ means $W + (-X)$, so $f(W - X) = f(W) - f(X)$.

For each $i \in V(\overrightarrow{C_n})$, define a function f_i like f, except that, in traversing W, we ignore all arcs except those $[x, y]$ for which $[\rho(x), \rho(y)] = [i, i+1]$ (arithmetic modulo n). Then $f = \sum_{i=0}^{n-1} f_i$. We claim that, if W is a *closed* walk, then $f_i(W) = f_j(W)$ for all i, j. To verify this, we show $f_i(W) = f_{i+1}(W)$ for each i. In traversing W, we may meet the fiber $\rho^{-1}(i+1)$ numerous times. Fig. 7 shows the three ways this can happen. Each of these possibilities contributes exactly the same amount to $f_i(W)$ and $f_{i+1}(W)$. The first contributes 1 to both $f_i(W)$ and $f_{i+1}(W)$ if the traversal is in the direction of the arrows, or -1 if it is against the arrows. The two cases on the right both contribute 0 to each. It follows that $f_i(W) = f_{i+1}(W)$.

For each closed walk W, we thus have $f(W) = \sum_{i=0}^{n} f_i(W) = nf_i(W)$, and this does not depend on i.

If $f_i(W)$ is odd for some closed walk W, then, for each i, W must have an odd number of arcs $[c, c']$ that project to $[i, i+1]$. (To see this, refer to Fig. 7. If there were an even number of such arcs, then $f_i(W)$ would be the sum of an even number of 1's and -1's, hence even.) To finish the proof, we show that such a W must exist.

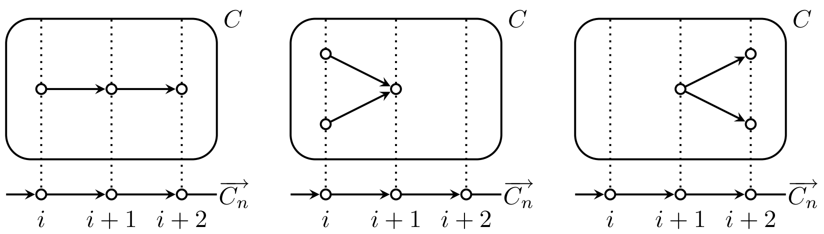

Fig. 7. The ways W can cross the fiber $\rho^{-1}(i+1)$. Each crossing contributes the same amount to $f_i(W)$ and $f_{i+1}(W)$.

To the contrary, suppose no such W exists. Then $f_i(W)$ is even for every closed walk W, and $f(W) = nf_i(W) = 2nd_W$ for some integer d_W (that depends on W). We are going to reach a contradiction by producing a homomorphism $\rho' : C \to \overrightarrow{C_{2n}}$, contradicting the fact that n is the largest integer for which there is a homomorphism $C \to \overrightarrow{C_n}$. Clearly it suffices to show how to construct such a homomorphism on each component of C, so henceforward we may assume C is connected.

Define a map $\rho' : V(C) \to V(\overrightarrow{C_{2n}})$ as follows. Fix a base point $c_0 \in V(C)$. Given any $c \in V(C)$, take a path P joining c_0 to c and set $\rho'(x) = f(P) \pmod{2n}$. This is well defined, for if P' is another such path, then $P - P'$ is a closed walk, so $f(P) - f(P') = f(P - P') = 2nd_{P+P'}$. Thus $f(P) = f(P') \pmod{2n}$.

To see that ρ' is a homomorphism, suppose $[c, c']$ is an arc of C. Take a path P from c_0 to c. Then $P + [c, c']$ is a path from c_0 to c'. We have $[\rho'(c), \rho'(c')] = [f(P), f(P + [c, c'])] = [f(P), f(P) + 1 \pmod{2n}] \in E(\overrightarrow{C_{2n}})$. $\square$

The next result is analogous to the converse direction of Theorem 4.

Proposition 4. *Suppose A, B and C are digraphs, and $n \geq 2$ is the largest integer for which C admits a homomorphism $\rho : C \to \overrightarrow{C_n}$. If $A \times C \cong B \times C$ then $B \cong A^{\pi}$, where π is a vertex of a null-walk of length n in $A!$.*

Proof. Let A, B, C and ρ be as stated. Suppose there is an isomorphism $\varphi : A \times C \to B \times C$. By Theorem 3, we assume that $\varphi(x, c) = (\varphi_c(x), c)$.

At this point the proof parallels that of Theorem 4. For any $[c, c'] \in E(C)$,

$$[x, y] \in E(A) \Longleftrightarrow [\varphi_c(x), \varphi_{c'}(y)] \in E(B), \tag{9}$$

$$[x, y] \in E(B) \Longleftrightarrow [\varphi_c^{-1}(x), \varphi_{c'}^{-1}(y)] \in E(A). \tag{10}$$

We now construct a null-walk of length n in $A!$. By Lemma 5, C has a closed walk W, which, for each i, has an odd number of arcs $[c, c']$ with $[\rho(c), \rho(c')] = [i, i+1]$. It is easy to confirm that this oddness criterion forces W to have a vertex c_0 that is neither a source nor sink in W. (That is, W has consecutive arcs of form $[c', c_0], [c_0, c'']$ or $[c_0, c''], [c', c_0]$.) Let c_0 be the initial (and terminal) vertex of W, and agree to traverse W with the orientation that gives consecutive arcs $[c', c_0], [c_0, c'']$. Also, arrange the indexing of $\overrightarrow{C_n}$ so that $\rho(c_0) = 0$. Then (locally) the fiber over the vertex $i + 1$ of $\overrightarrow{C_n}$ is as in Fig. 6.

We label the vertices of W as we did those of P in the proof of Theorem 4: for a given $i \in V(\overrightarrow{C_n})$, suppose $[c, c'] \in E(W)$ is the jth arc of W for which $\rho([c, c']) = [i, i+1]$. Label c with the map $\lambda_{ij} = \varphi_c$, and label c' with the map $\mu_{ij} = \varphi_{c'}$. (See Fig. 6, but replace $\overrightarrow{P_n}$ with $\overrightarrow{C_n}$.)

Thanks to Lemma 5, given i, there is an odd number of labels $\lambda_{i1}, \lambda_{i2}, \lambda_{i3}, \ldots, \lambda_{i\ell_i}$, and the same (odd) number of labels $\mu_{i1}, \mu_{i2}, \mu_{i3}, \ldots, \mu_{i\ell_i}$. For each $0 \leq i \leq n-1$, put

$$L_i = \lambda_{i1}\lambda_{i2}^{-1}\lambda_{i3}\lambda_{i4}^{-1} \cdots \lambda_{i\ell_i}$$

$$M_i = \mu_{i1}\mu_{i2}^{-1}\mu_{i3}\mu_{i4}^{-1} \cdots \mu_{i\ell_i}.$$

The terms λ_{ij} and μ_{ij} that appear in these expressions are bijections $\lambda_{ij} = \varphi_c$ and $\mu_{ij} = \varphi_{c'}$ for some arc $[c, c'] \in E(W) \subseteq E(C)$. Applying Equivalences (9) and (10) successively (an odd number of times) yields

$$[x, y] \in E(A) \Longleftrightarrow [L_i(x), M_i(y)] \in E(B).$$

Just as in the proof of Theorem 4, we get $M_i = L_{i+1}$, but this time the index arithmetic can be done modulo n. Then

$$[x, y] \in E(A) \iff [L_i(x), L_{i+1}(y)] \in E(B),$$

and from this:

$$[x, y] \in E(A) \iff [L_i^{-1}L_{i+1}(x), L_{i+1}^{-1}L_{i+2}(y)] \in E(B),$$

for each $0 \le i \le n-1$, where the index arithmetic is done modulo n.

Therefore we have the following directed closed walk of length n in $A!$:

$$[L_0^{-1}L_1,\ L_1^{-1}L_2], [L_1^{-1}L_2,\ L_2^{-1}L_3], [L_2^{-1}L_3,\ L_3^{-1}L_4], \ldots, [L_{n-1}^{-1}L_0,\ L_0^{-1}L_1].$$

Multiplying arcs, we see that this is a null-walk with initial vertex $L_0^{-1}L_1$.

To finish the proof, note that $L_0 : A^{L_0^{-1}L_1} \to B$ is an isomorphism, by the same argument used in the last paragraph of the proof of Theorem 4. □

We caution that if $A \times C \cong A^\sigma \times C$, then Proposition 4 does not imply that σ is necessarily on a null-walk in $A!$ of length n; rather it implies $A^\sigma \cong A^\pi$ for some π on such a null-walk. (By Proposition 2, this means that the $E(A!)$-orbit of σ meets a null-walk of length n.) For a simple example of this phenomenon, let $C = \overrightarrow{C_2}$, so $n = 2$. Let $A = \overrightarrow{C_3}$, whose factorial appears in the bottom of Fig. 5. Using the vertex labeling of that figure, put $\sigma = (012)$. The reader may check that $A \times C \cong A^\sigma \times C$, but σ is not on a null-walk of length 2. However, the orbit of σ meets the identity, which is on a null-walk of length 2 (i.e., the loop at id traversed twice).

We now adapt the previous two propositions to our final theorem. Theorem 1 implies that each component of a zero divisor C admits a homomorphism into a directed cycle of prime length, and, by our previous discussion, each component is a zero divisor of Type P or C. Let C be a disjoint union $C = C^1 + C^2 + \cdots + C^k$ of connected zero divisors of Type C, each of which admits a homomorphism $C^i \to \overrightarrow{C_{n_i}}$, where n_i is the largest such integer. We argue that $A \times C \cong B \times C$ if and only if $B \cong A^\pi$, where the $E(A!)$ orbit of π meets null-walks of lengths $n_1, n_2, \ldots, n_k$.

If indeed the orbit of π meets null-walks of lengths $n_1, n_2, \ldots, n_k$, then Propositions 2 and 3 imply that $A \times C^i \cong B \times C^i$ for $1 \le i \le k$. Because the direct product distributes over disjoint union, we have

$$\begin{aligned} A \times C &= A \times C^1 + A \times C^2 + \cdots + A \times C^k, \\ B \times C &= B \times C^1 + B \times C^2 + \cdots + B \times C^k. \end{aligned} \tag{11}$$

(This is equality, not mere isomorphism.) It follows that $A \times C \cong A \times B$.

Conversely suppose there is an isomorphism $\varphi : A \times C \to B \times C$, which we may assume to have form $(x, c) \mapsto (\varphi_c(x), c)$. Combining this with Eqs. (11), it follows that φ restricts to an isomorphism $A \times C^i \cong B \times C^i$ for each i. Propositions 2 and 3 imply that $B \cong A^\pi$, where the $E(A!)$ orbit of π meets null-walks of lengths $n_1, n_2, \ldots, n_k$.

In fact, in the above reasoning, there is no harm in adding to C some components of Type P, for a directed walk of length $n-2$ in $A!$ (recall Theorem 4) can be found in any null-walk by "wrapping around" to the extent needed. Combining the above discussion with Theorem 4, Propositions 3 and 4, and adapting the discussion preceding Corollary 3, we get the following theorem. It covers all zero divisors not addressed in Section 6.

Theorem 6. *Suppose C is an arbitrary zero divisor of Type C, so it is a disjoint union*

$$C = C^1 + C^2 + \cdots + C^k + P^1 + P^2 + \cdots + P^\ell$$

of connected zero divisors, where each C^i is of Type C and each P^i is of Type P. (And possibly no P^i are present.) For each index $1 \le i \le k$, let n_i be the largest integer for which there is a homomorphism $C^i \to \overrightarrow{C_{n_i}}$.

Then $A \times C \cong B \times C$ if and only if $B = A^\pi$, where the $E(A!)$ orbit of $\pi \in V(A!)$ meets null-walks of lengths $n_1, n_2, \ldots, n_k$.

Let Υ be the set of all such π. Take a collection $\pi_1, \pi_2, \ldots, \pi_q \in \Upsilon$ of representatives of all orbits of the $E(A!)$ action on Υ. For A and C, the set of distinct digraphs B for which $A \times C \cong B \times C$ is $B = A^{\pi_1}, A^{\pi_2}, \ldots, A^{\pi_q}$. If $q = 1$, then cancellation holds.

8. Applications to graphs

The questions we have posed can also be asked of graphs, that is, of symmetric digraphs: if A and B are graphs, find all graphs B for which $A \times C \cong B \times C$. When does cancellation hold? Of course our results apply this situation, but the additional structure leads to simplifications and unexpected twists. We now examine this.

To begin, observe that the factorial of a graph A is symmetric: $[\alpha, \beta] \in E(A!)$ if and only if $[\alpha(x), \beta(y)] \in E(A)$ precisely when $[x, y] \in E(A)$, if and only if $[\beta(y), \alpha(x)] \in E(A)$ precisely when $[y, x] \in E(A)$, if and only if $[\beta, \alpha] \in E(A!)$. In summary, if A is a graph, then its factorial is also a graph.

Next, we claim that a graph C is a zero divisor if and only if it is bipartite. Suppose C is a zero divisor. If C has an edge, then it has no homomorphism into any directed path or cycle that is not already a graph. By Theorem 1, C has a homomorphism into the graph $\overrightarrow{C}_2$, that is, C is bipartite. Conversely, let C be bipartite, so it has a homomorphism into $\overrightarrow{C}_2$. Theorem 1 implies it is a zero divisor *in the class of digraphs*. That is, there are non-isomorphic *digraphs* A, B with $A \times C \cong B \times C$. Since A and B should be *graphs*, we are not quite done. But simply take $A = \overrightarrow{C}_2$ and $B = 2\overrightarrow{C}_1$. Then $A \times C \cong 2C \cong B \times C$, so C is a zero divisor in the class of graphs, completing the claim.

Note that a bipartite graph C (with at least one edge) is necessarily a zero divisor of Type C. Given such a C and a graph A, the previous section implies that $A \times C \cong B \times C$ if and only if $A \cong B^{\pi}$, where π is on a null-walk of length 2 in $A!$. Such a walk necessarily has form $[\pi, \pi^{-1}][\pi^{-1}, \pi]$, and $[x, y] \in E(A)$ if and only if $[\pi(x), \pi^{-1}(y)] \in E(A)$.

A permutation π of $V(A)$ satisfying $[x, y] \in E(A) \iff [\pi(x), \pi^{-1}(y)] \in E(A)$ is called an *anti-automorphism* of A in [1], and we adopt that term here. Denote by $\mathrm{Ant}(A)$ the set of anti-automorphisms of A, so $\mathrm{Ant}(A)$ is the set of vertices of $A!$ on null-walks of length 2; it contains the identity and is closed under inverses, though not composition (it is not a group).

The $E(A!)$ action on $V(A!)$ is stable on $\mathrm{Ant}(A) \subseteq V(A!)$: indeed, take $\pi \in \mathrm{Ant}(A)$. If $[\alpha, \beta] \in E(A!)$, we have $[\beta, \alpha] \in E(A!)$ (because $A!$ is a graph). The definitions imply $[x, y] \in E(A) \iff [\alpha\pi\beta^{-1}(x),\ \beta\pi^{-1}\alpha^{-1}(y)] \in E(A) \iff [\alpha\pi\beta^{-1}(x),\ (\alpha\pi\beta^{-1})^{-1}(y)] \in E(A)$. Thus $\alpha\pi\beta^{-1} = [\alpha, \beta] \cdot \pi \in \mathrm{Ant}(A)$.

Finally, we caution that if A is a graph and $\pi \in V(A!)$ is arbitrary, then although A^{π} is a digraph, it need not be a graph. However, if A is a graph, then A^{π} is a graph if and only if $\pi \in \mathrm{Ant}(A)$. The simple proof is omitted.

Combining these considerations with Theorem 6 yields our final result.

Theorem 7. *Suppose A, B and C are graphs and C is a zero divisor (that is, bipartite). Then $A \times C \cong B \times C$ if and only if $B \cong A^{\pi}$ for some $\pi \in \mathrm{Ant}(A)$.*

Given A and C, the set of all distinct graphs X for which $A \times C \cong X \times C$ can be found as follows. Take distinct representatives $\pi_1, \pi_2, \ldots, \pi_q$ of the orbits of the $E(A!)$ action on $\mathrm{Ant}(A)$. Then $X = A^{\pi_1}, A^{\pi_2}, \ldots, A^{\pi_q}$.

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