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ART DEPARTMENT

Topological Envelopes

—Richard H. Hammack

The mail arrives. Pick up a letter, and behold a topological curiosity. The envelope divides space into two regions: You are outside, and a letter is inside. The envelope is a sphere!

Another envelope contains a bill. It has a window that reveals your address, which is printed on the bill. The envelope is a sphere with a disk removed!

So it was some years ago that I wondered what other surfaces can be sent through the mail.

Figure 1. *Standard envelopes: A sphere (left), and a sphere minus a disk (right).*

The window offered a wealth of possibilities. One of my first constructions was a Klein bottle (Figure 2) having the form of a flat tube with a window, folded so that one end of the tube passes through the window and glues to the other end. There is no need to open this envelope, because it has no interior! I use it like a postcard, with a message on the back.

Like all of my envelopes, the Klein bottles begin life as a LaTeX file that is typeset, printed, cut, folded and glued. But it's not really finished until it's mailed to a friend or colleague. I see my envelopes as being closely aligned with printmaking, with the return address and postmarked stamp standing in for a signature and edition number.

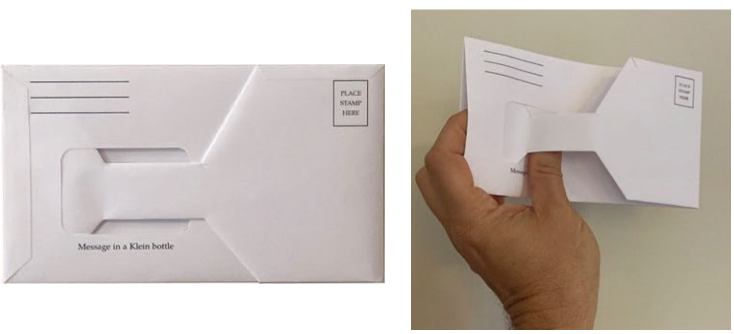

Figure 2. *Message in a Klein Bottle. Printed on a single legal-size sheet of paper, it is then cut, folded, glued, … and mailed!*

Figure 3. *The projective plane (left) and Klein bottle (right). Each has exactly one hole, bounded by a black contour.*

Figure 3 shows two envelopes obtained by attaching folded strips to a window. The first is a projective plane (with a hole). The second is another realization of a Klein bottle (with a hole). Each strip is a portion of a Möbius band that gives passage from the "outside" of the envelope to the "inside." In theory, one could add n strips in sequence around the window, resulting in a sphere with n cross-caps (and one hole). By the classification theorem for compact surfaces, it's therefore possible to make a one-holed envelope that is any non-orientable, connected, compact surface we desire!

But the placement of the strips is significant. The envelope in Figure 4 has a hexagonal window and three interlocking folded strips. It has three holes, bounded by red, green and blue lines. But it is *not* a sphere with three cross-caps, as the reader can verify by computing its Euler characteristic (after mentally capping the three holes with hexagons). The Euler characteristic is –1, so the envelope is a projective plane. It is, in fact, a flat variant of Boy's surface. Werner Boy discovered this fascinating surface in 1901 when David Hilbert (his dissertation advisor) asked him to prove that the projective plane cannot be immersed in three-dimensional space. Instead, Boy proved the opposite by producing a beautiful immersion with 3-fold symmetry.

Not all of my envelopes are non-orientable. Figure 5 shows two toroidal examples.

If you are interested in the intersection of math and art, consider attending a Bridges conference! Bridges is the premier organization devoted to mathematics and the arts

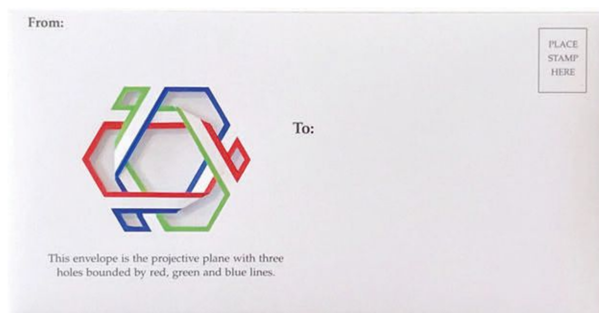

Figure 4. *Boy's surface (with three holes). Imagine widening the three bands: the three holes would narrow into three slits through which the paper would pass through itself. But I designed the envelope with holes (rather than slits) so the viewer can look inside and better comprehend the structure.*

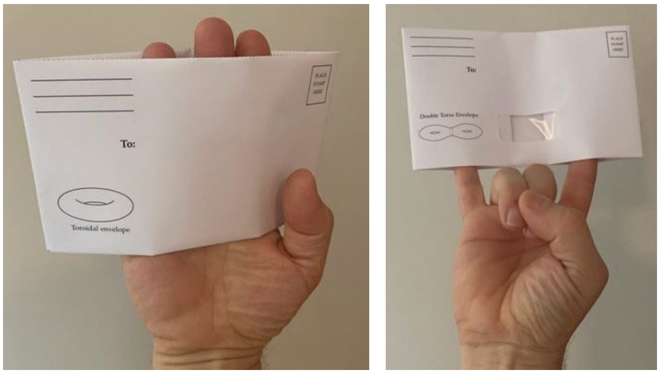

Figure 5. *The toroidal envelope (left) has a flap at the top that opens to accept a cylindrical letter. The double torus (right) has a transparent plastic window to better reveal its topology.*

(bridgesmathart.org). Recent conference venues have included Stockholm, Linz, Helsinki, and Halifax, Nova Scotia. The summer 2024 conference will be at Virginia Commonwealth University, in Richmond, Virginia. Figure 6 shows my design for the (one-sided) conference postcard. I hope to see you there!

Figure 6. *One-sided postcard for the 2024 Bridges conference.*

Richard Hammack *is a professor of mathematics at Virginia Commonwealth University. He is the author of* Book of Proof, *a popular open-source proofs textbook, and coauthor of a research monograph on graph products. He works mostly in combinatorics and graph theory and has recently begun making mathematical art. His envelopes won the prize for best painting/photograph/print at the 2023 JMM Mathematical Art Exhibition. They will also be featured in the exhibit* Seeing the Unseen: Math and Art *at the Wignall Museum of Contemporary Art at Chaffey College in Rancho Cucamonga, California, January 8–March 9, 2024.*

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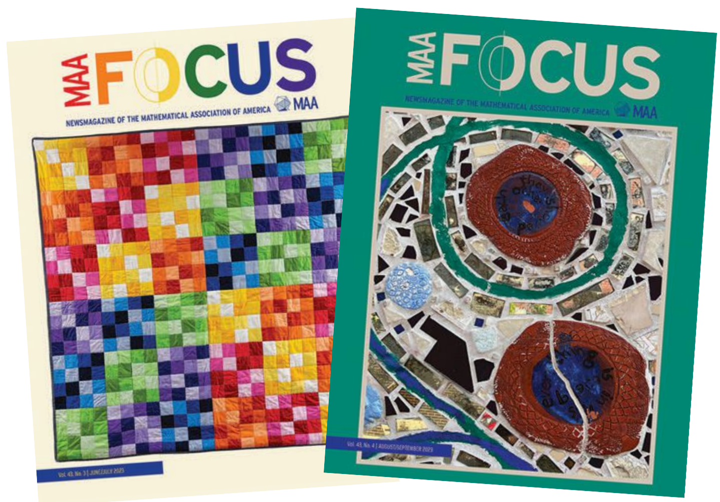

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