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Minimum cycle bases of direct products of graphs with cycles

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Abstract

We construct a minimum cycle basis for the direct product $G \times C_q$ where G is a connected non-bipartite graph, C_q is an odd cycle and $G \times C_q$ is triangle-free. These bases are expressed in terms of the cycle structure of the symmetric digraph on G.

Keywords: Graph direct product, minimum cycle bases, digraphs.

Math. Subj. Class.: O5C38

1 Introduction

In the article *Minimum Cycle Bases of Product Graphs* [7], W. Imrich and P. Stadler construct minimum cycle bases for Cartesian and strong products of graphs, in terms of minimum cycle bases of the factors. F. Berger [1] and M. Jaradat [8] solve the same problem for the lexicographical product. The corresponding construction for the direct product appears to be extremely complex. The problem has been solved for the special cases where both factors are bipartite [4] or complete graphs [2, 5]. The authors of the present paper have an outline of a construction for a minimum cycle basis of the product $G \times H$ where both factors are arbitrary connected non-bipartite graphs (and at least one factor is triangle-free), but the proofs are too long to be of much interest. In this paper we offer a scaled-back version of the problem. We describe minimum cycle bases for $G \times C_q$ where G is a connected non-bipartite graph and C_q is the odd cycle on q vertices.

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We begin with a brief review of the preliminaries. The **edge space** $\mathcal{E}(G)$ of a simple graph $G = (V(G), E(G))$ is the power set of its edges $E(G)$ endowed with the structure of a vector space over the two-element field $\mathbb{F}_2 = \{0, 1\}$. Addition in $\mathcal{E}(G)$ is symmetric difference of sets, and zero is the empty set. To simplify notation, we blur the distinction between subgraphs K of G and their edge set $E(K) \in \mathcal{E}(G)$. Thus, if J and K are subgraphs, an expression such as $J + K$ means $E(J) + E(K)$, with the operation taking place in $\mathcal{E}(G)$. With this convention, we regard $E(G)$ as the set of all connected one-edge subgraphs of G, so $E(G)$ a basis for $\mathcal{E}(G)$. Similarly, the **vertex space** $\mathcal{V}(G)$ of G is the power set of $V(G)$ endowed with a vector space structure over $\mathbb{F}_2$. (Addition is symmetric difference.) Bending the notation slightly (as was done for the edge space) we regard $V(G)$ as a basis for $\mathcal{V}(G)$.

The **cycle space** of G, denoted $\mathcal{C}(G)$, is the subspace of $\mathcal{E}(G)$ consisting of the edge sets $E(K)$ of Eulerian subgraphs K of G, that is, subgraphs having no vertex of odd degree. (See [3], Proposition 1.9.2). Elements of $\mathcal{C}(G)$ are called **cycles**. We call a cycle a **simple cycle** if it is connected and each vertex has degree 2. The simple cycle with p edges is denoted C_p. The dimension of $\mathcal{C}(G)$ is the (first) Betti number $\beta(G) = |E(G)|-|V(G)|+c$, where c is the number of components of G ([3], Theorem 1.9.6). A graph homomorphism $g : G \to H$ induces a linear map $g^* : \mathcal{E}(G) \to \mathcal{E}(H)$ defined on the basis $E(G)$ as $g^*(vw) = g(v)g(w)$. It is easy to check that g^* restricts to a linear map $g^* : \mathcal{C}(G) \to \mathcal{C}(H)$.

A basis $\mathcal{B}$ of $\mathcal{C}(G)$ is called a **cycle basis of G**, and its **length** is $\ell(\mathcal{B}) = \sum_{C\in\mathcal{B}} |C|$. Among all cycle bases of G, one with smallest possible length is called a **minimum cycle basis**, or **MCB**. It is easy to check that every MCB contains only simple cycles.

The cycle space is a weighted matroid where each element C has weight $|C|$. Hence the Greedy Algorithm [10] always terminates with an MCB. (I.e. begin with $\mathcal{M} = \emptyset$; then append shortest cycles to it, maintaining independence of $\mathcal{M}$, until no further shortest cycles can be appended; then append next-shortest cycles, maintaining independence, until no further such cycles can be appended; and so on, until $\mathcal{M}$ is a maximal independent set. Then $\mathcal{M}$ is an MCB.)

Here is our primary criterion for determining if a cycle basis is an MCB.

Proposition 1.1. *A cycle basis $\mathcal{B} = \{B_1, B_2, \ldots, B_{\beta(G)}\}$ for a graph G is an MCB if and only if every $C \in \mathcal{C}(G)$ is a sum of basis elements whose lengths do not exceed $|C|$.*

Proof. Suppose $\mathcal{B}$ is an MCB, but there is a cycle $C = \sum_{k=1}^{\beta(G)} b_k B_k$ (each b_k is in $\mathbb{F}_2$) and $|C| < |B_k|$ for some k with $b_k \neq 0$. Then we can exchange basis element B_k for C and obtain a basis with smaller total length than $\mathcal{B}$, contradicting minimality. Conversely, suppose $\mathcal{B}$ is not an MCB. Assume that the elements B_1, $B_2, \ldots$ are arranged in order of increasing length. Since the greedy algorithm cannot terminate with basis $\mathcal{B}$, there must be an element B_p for which the set $\{B_1, B_2, \ldots, B_{p-1}\}$ can be extended to an independent set $\{B_1, B_2, \ldots, B_{p-1}, C\}$ with $|C| < |B_p|$. Necessarily then, $C = \sum_{k=1}^{\beta(G)} b_k B_k$ with $b_i \neq 0$, for some $p \leq i \leq \beta(G)$, and $|C| < |B_i|$. □

The *direct product* of graphs G and H is the graph $G \times H$ whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edges are $(u, x)(v, y)$ where $uv \in E(G)$ and $xy \in E(H)$. We quickly mention a few standard facts; the reader requiring more background is referred to [6]. Suppose G and H are connected. Then $G \times H$ is connected if and only if one of G and H has an odd cycle. Otherwise, if G and H are both bipartite then $G \times H$ has exactly two components. As $G \times H$ has $|V(G)| \cdot |V(H)|$ vertices and

$2|E(G)| \cdot |E(H)|$ edges we have $\beta(G \times H) = 2|E(G)| \cdot |E(H)| - |V(G)| \cdot |V(H)| + 1$ whenever G and H are connected and one has an odd cycle.

Note that the standard projection maps $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ induce projection operators $\pi_G^* : \mathcal{C}(G \times H) \to \mathcal{C}(G)$ and $\pi_H^* : \mathcal{C}(G \times H) \to \mathcal{C}(H)$.

For any simple cycle C_p, we put $V(C_p) = \mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$ and we agree that the edges of C_p join i to $i+1$ for each $i \in \mathbb{Z}_p$. Thus an arbitrary edge is written $i(i+1)$, which we take care not to confuse with multiplication.

2 An MCB for $G \times K_2$

To motivate our approach for constructing MCB's of $G \times C_q$, we now examine the problem of constructing an MCB for $G \times K_2$, where G is an arbitrary connected graph and K_2 is the complete graph on 2 vertices. We put $V(K_2) = \{0, 1\}$ and denote its edge as 01.

We remark at the onset that an MCB of G typically bears little resemblance to an MCB of $G \times K_2$. For example, consider Figure 1. The factor G consists of a pentagon that shares a vertex with a triangle. Obviously, an MCB for G consists of just two cycles, the pentagon and the triangle. But an MCB of $G \times K_2$ consists of two 8-gons (shown solid and dashed) plus the 6-gon over the triangle. The two 8-gons do not seem in any way related to any single element of the MCB for G. Our main task in this section is to uncover how an MCB for $G \times K_2$ is related to the structure of the factor G.

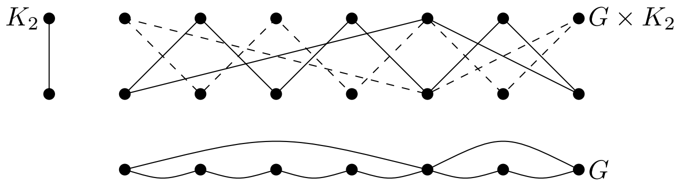


Figure 1: An example of $G \times K_2$

Our basic approach is to employ not the cycle structure of G, but instead the cycle structure of the symmetric digraph on G. Any graph G can be identified with a symmetric digraph $\overleftrightarrow{G}$ obtained by replacing each edge xy of G with an arc $\overrightarrow{xy}$ directed from x to y and an arc $\overrightarrow{yx}$ directed from y to x. Thus $\overleftrightarrow{G}$ has vertex set $V(\overleftrightarrow{G}) = V(G)$ and arc set $E(\overleftrightarrow{G}) = \{\overrightarrow{xy}, \overrightarrow{yx} : xy \in E(G)\}$. We define an **anti-cycle** to be a sub-digraph of $\overleftrightarrow{G}$, each vertex of which has even in-degree and even out-degree. Figure 2 shows an anti-cycle for the factor G from Figure 1. Let $\mathcal{A}(\overleftrightarrow{G})$ denote the set of anti-cycles in $\overleftrightarrow{G}$. In what follows we describe how $\mathcal{A}(\overleftrightarrow{G})$ is naturally identified with $\mathcal{C}(G \times K_2)$, and how "minimum anti-cycle bases" of $\mathcal{A}(\overleftrightarrow{G})$ correspond to MCB's of $G \times K_2$.

Figure 2: An anti-cycle in G

Let $\mathcal{E}(\overleftrightarrow{G})$ denote the power set of $E(\overleftrightarrow{G})$ endowed with a vector space structure over

$\mathbb{F}_2$. (Addition is symmetric difference, etc.) As usual, to keep the notation under control we identify elements of $\mathcal{E}(\overleftrightarrow{G})$ with sub-digraphs of $\overleftrightarrow{G}$, so, for example, an element $\{\overrightarrow{xy}\} \in \mathcal{E}(\overleftrightarrow{G})$ is written simply as $\overrightarrow{xy}$. With this convention, the set $E(\overleftrightarrow{G})$ is a basis for $\mathcal{E}(\overleftrightarrow{G})$, which has dimension $2|E(G)|$.

Define a linear map $\pi : \mathcal{E}(G \times K_2) \to \mathcal{E}(\overleftrightarrow{G})$ which acts on the basis $E(G \times K_2)$ as $\pi((x, 0)(y, 1)) = \overrightarrow{xy}$. This is a vector space isomorphism because it sends the basis $E(G \times K_2)$ of $\mathcal{E}(G \times K_2)$ injectively onto the basis $E(\overleftrightarrow{G})$ of $\mathcal{E}(\overleftrightarrow{G})$. A moment's thought confirms that π restricts to an isomorphism $\pi : \mathcal{C}(G \times K_2) \to \mathcal{A}(\overleftrightarrow{G})$.

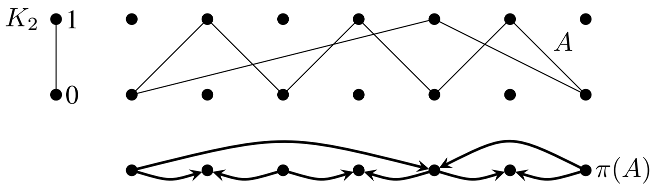


Figure 3: How anti-cycles in G correspond to cycles in $G \times K_2$

Let the **length** $|C|$ of an element $C \in \mathcal{C}(\overleftrightarrow{G})$ be the number of arcs in C. Observe that $|A| = |\pi(A)|$ for every $A \in \mathcal{C}(G \times K_2)$. Thus π is a length-preserving isomorphism from $\mathcal{C}(G \times K_2)$ to $\mathcal{A}(\overleftrightarrow{G})$.

In analogy with the definition of an MCB, we define a **minimum anti-cycle basis** of $\overleftrightarrow{G}$ to be a basis $\{B_1, B_2, \ldots, B_{\beta(G\times K_2)}\}$ for $\mathcal{A}(\overleftrightarrow{G})$ in which the total length $\sum |B_i|$ has the smallest possible value. The remarks above show that π gives an exact correspondence between minimum anti-cycle bases of $\overleftrightarrow{G}$ and minimum cycle bases of $G \times K_2$. This leads to the following result linking MCBs of $G \times K_2$ to the structure of G.

Proposition 2.1. *Let $\pi : \mathcal{C}(G \times K_2) \to \mathcal{A}(\overleftrightarrow{G})$ be the isomorphism defined by the rule $\pi((x, 0)(y, 1)) = \overrightarrow{xy}$. Then $\{B_1, B_2, B_3, \ldots, B_{\beta(G\times K_2)}\}$ is a minimum anti-cycle basis for $\overleftrightarrow{G}$ if and only if the set $\{\pi^{-1}(B_1), \pi^{-1}(B_2), \pi^{-1}(B_3), \ldots, \pi^{-1}(B_{\beta(G\times K_2)})\}$ is an MCB for $G \times K_2$.*

We do not address the question of how to compute a minimum anti-cycle basis, since our main goal is to express an MCB of $G \times K_2$ in terms of invariants of the factors, and Proposition 2.1 accomplishes this in the sense that the structure of $\overleftrightarrow{G}$ is inherent in the structure of G. We'll later see that other cycle structures in $\overleftrightarrow{G}$ can be used to construct MCB's in $G \times C_q$.

3 The Diamond Space

Our ultimate goal is to obtain MCB's of $G \times C_q$, where q is odd and one of the factors is triangle-free. In this situation $G \times C_q$ is triangle-free, for if K were a triangle in $G \times C_q$, then $\pi_G(K)$ and $\pi_{C_q}(K)$ would be a triangles in G and C_q. Hence the shortest cycles in $G \times C_q$ have length at least four. Since—by the greedy algorithm—an MCB must contain a maximal independent set of shortest cycles, we are especially concerned with forming a maximal independent set of squares. This section deals with certain squares in $G \times C_q$ called diamonds.

If $Q = abc$ and $R = def$ are paths of length 2 in graphs G and H, respectively, let QR denote the 4-cycle $(a, e)(b, f)(c, e)(b, d)(a, e)$ in $G \times H$, as illustrated in Figure 4. Such a subgraph QR is called a **diamond** in $G \times H$, and the subspace $\mathcal{D}(G \times H) \subseteq \mathcal{C}(G \times H)$ spanned by all diamonds is called the **diamond space** of $\mathcal{C}(G \times H)$. In what follows we construct a basis of diamonds for $\mathcal{D}(G \times C_q)$. Our construction involves certain spaces of paths of length 2 in the factors. Let us agree to call a path abc of length 2 in a graph a P_2 **centered at** b in the graph. The P_2's of C_q are all of form $(i-1)i(i+1)$ for $i \in \mathbb{Z}_q$.

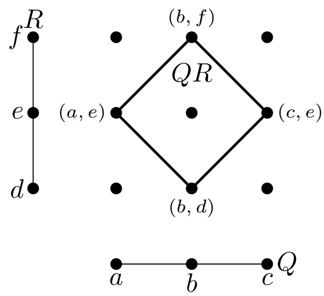


Figure 4: A diamond QR

Given a vertex $b \in V(G)$, let $S(b)$ be the set of edges of G that are incident with b. (We may think of $S(b)$ as a subgraph of G, i.e. the "star" at b.) Let $\mathcal{P}(G, b) = \{X \subseteq E(S(b)) : |X| \text{ is even}\}$. Note that $\mathcal{P}(G, b)$ is the subspace of $\mathcal{E}(S(b))$ spanned by the P_2's in G centered at b, so we call it the P_2 **space of** G **at** b. Observe that $\mathcal{P}(G, b)$ is the kernel of the surjective linear map $\mathcal{E}(S(b)) \to \mathbb{F}_2$ given by $X \mapsto |X| \pmod 2$, so we have $\dim(\mathcal{P}(G, b)) = \dim(\mathcal{E}(S(b))) - 1 = \deg_G(b) - 1$. If the neighborhood of b is labeled as $N_G(b) = \{x_0, x_1, x_2, \ldots, x_{\deg(b)-1}\}$, then the set of P_2's $\mathcal{B} = \{x_0bx_1, x_0bx_2, \ldots, x_0bx_{\deg(b)-1}\}$ is a basis for $\mathcal{P}(G, b)$. A basis for $\mathcal{P}(G, b)$ that consists entirely of P_2's is called a P_2 **basis for** $\mathcal{P}(G, b)$. A P_2 basis for $\mathcal{P}(G, b)$ of the form $\mathcal{B}$ above (for which some edge x_0b belongs to every element of the basis) is called a **standard** P_2 **basis with common edge** x_0b. A version of the next lemma was proved in [9]. For completeness we include a separate proof here.

Lemma 3.1. *Suppose T is a tree with at least two edges, and for each $t \in V(T)$ let $\mathcal{T}_t$ be a P_2 basis for $\mathcal{P}(T, t)$. Then the set of diamonds $\mathcal{D} = \{Q[(i-1)i(i+1)] : Q \in \mathcal{T}_t, t \in V(T), i \in V(C_q)\}$ is linearly independent in $\mathcal{C}(T \times C_q)$.*

Proof. We use induction on $|E(T)|$. First suppose $|E(T)| = 2$, so $T \cong P_2$. Label its vertices so that $T = abc$. Then $\mathcal{D} = \{[abc][(i-1)i(i+1)] : i \in V(C_p)\}$. Notice that if $i \neq j$ then diamonds $[abc][(i-1)i(i+1)]$ and $[abc][(j-1)j(j+1)]$ have no edges in common, since each edge in $[abc][(i-1)i(i+1)]$ has either (a, i) or (c, i) as an endpoint, but no edge of $[abc][(j-1)j(j+1)]$ has these endpoints. Thus the elements of $\mathcal{D}$ are pairwise edge-disjoint, so $\mathcal{D}$ is linearly independent.

Now suppose the statement is true for any tree T with fewer than $n \geq 3$ edges, and suppose $|E(T)| = n$. For each $x \in V(T)$, let $\mathcal{T}_x$ be a P_2 basis for $\mathcal{P}(T, x)$. Observe that there is an edge of T that belongs to exactly one P_2 in $\mathcal{T} = \bigcup_{x\in V(T)} \mathcal{T}_x$. To see this, note that since $|\mathcal{T}_x| = \deg_T(x) - 1$ and the sets $\mathcal{T}_x$ are pairwise disjoint, we have $|\mathcal{T}| = \sum_{x\in V(T)}(\deg_T(x) - 1) = 2|E(T)| - |V(T)| = |E(T)| - 1$. Since each element of $\mathcal{T}$ has two edges, then on average each edge of T belongs to $\frac{2(|E(T)-1)}{|E(T)|} < 2$ elements in $\mathcal{T}$. Thus some edge of T belongs to fewer than two P_2's in $\mathcal{T}$. But each edge in T belongs to at least one element of $\mathcal{T}$, so some edge of T belongs to exactly one P_2 in $\mathcal{T}$, as claimed.

Notice that if st is an edge that belongs to just one P_2 in $\mathcal{T}$, then one of s or t has degree 1, for otherwise each of $\mathcal{T}_s$ and $\mathcal{T}_t$ have (distinct) elements that contains st. Thus there is an $st \in E(T)$, with $\deg_T(s) = 1$, such that st meets exactly one diamond stu in $\mathcal{T}$. Let $T' = T - s$. (i.e. T' is T with vertex s and edge st removed.) Now, for each $x \in V(T') - \{t\}$ the set $\mathcal{T}_x$ remains a P_2 basis for $\mathcal{P}(T', x)$, and $\mathcal{T}_t - \{stu\}$ is a P_2 basis for $\mathcal{P}(T', t)$. Let $\mathcal{T}'_x = \mathcal{T}_x$ for $x \in V(T') - \{t\}$ and $\mathcal{T}'_t = \mathcal{T}_t - \{stu\}$.

By induction, the set $\mathcal{D}' = \{Q[(i-1)i(i+1) : Q \in \mathcal{T}'_x, x \in V(T'), i \in V(C_q)\}$ is linearly independent in $\mathcal{C}(T' \times C_q) \subseteq \mathcal{C}(T \times C_q)$. To complete the proof we need to show that the set $\mathcal{D} = \{Q[(i-1)i(i+1)] : Q \in \mathcal{T}_x, x \in V(T), i \in V(C_q)\}$ is linearly independent. Notice that $\mathcal{D}$ is a disjoint union $\mathcal{D} = \mathcal{D}' \cup \mathcal{D}''$, where $\mathcal{D}'' = \{[stu][(i-1)i(i+1)] : i \in V(C_q)\}$. By induction, the set $\mathcal{D}''$ is linearly independent in $\mathcal{C}(stu\times C_q) \subseteq \mathcal{C}(T \times C_q)$. To show that $\mathcal{D}$ is independent, it suffices to show that $\text{span}(\mathcal{D}')\cap\text{span}(\mathcal{D}'') = \{0\}$. Thus suppose $W \in \text{span}(\mathcal{D}') \cap \text{span}(\mathcal{D}'')$. Notice that $\mathcal{D}''$ consists of exactly q diamonds, all of form $[stu][(i-1)i(i+1)]$, and no two of which share an edge. Thus since $W \in \text{span}(\mathcal{D}'')$, W is the (possibly empty) edge-disjoint union of diamonds from $\mathcal{D}''$. But also $W \in \text{span}(\mathcal{D}')$, so W can't have any edges of form $(s, i)(t, i \pm 1)$. Since every diamond in $\mathcal{D}''$ has such edges, we conclude $W = 0$. □

One may wonder if the tree T in Lemma 3.1 can be replaced with an arbitrary connected graph G. If $\mathcal{G}_x$ is a P_2 basis for $\mathcal{P}(G, x)$ for each $x \in V(G)$, will the set of diamonds $\mathcal{D} = \{Q[(i-1)i(i+1)] : Q \in \mathcal{G}_x, x \in V(G), i \in V(C_q)\}$ be linearly independent in $\mathcal{C}(G\times C_q)$? Though the answer is "no," just a small number of diamonds need to be removed to make the set independent. The details are outlined in the following construction for a linearly independent set of diamonds in $\mathcal{D}(G \times C_q)$. We will only prove independence here, but later we'll see that the set is actually a basis for $\mathcal{D}(G \times C_q)$.

Construction 3.2. (A basis of diamonds for $\mathcal{D}(G \times C_q)$, with G connected and q odd.)
Let T be a spanning tree of G.
Let $E(G) - E(T) = \{b_1c_1,\ b_2c_2,\ \ldots,\ b_{\beta(G)}c_{\beta(G)}\}$.
For each $x \in V(G)$ let $\mathcal{G}_x$ be a standard P_2 basis for $\mathcal{P}(G, x)$ whose common edge belongs to T.
For each $1 \le i \le \beta(G)$ let $a_ib_ic_i$ denote the element of $\mathcal{G}_{b_i}$ containing the edge b_ic_i.
Let $\mathcal{D} = \{Q[(i-1)i(i+1)] : Q \in \mathcal{G}_x, x \in V(G), i \in V(C_q)\} - \{[a_ib_ic_i][012] : 1 \le i \le \beta(G)\}$.
Then *$\mathcal{D}$ is a linearly independent set of diamonds. Moreover, $|\mathcal{D}| = \beta(G\times C_q) - \beta(G) - 1$.*

Proof. We use induction on $\beta(G)$. If $\beta(G) = 0$, then $G = T$. In this case $\mathcal{D}$ is linearly independent by Lemma 3.1.

Now assume the statement is true for all G with $\beta(G) < n$, for some integer $n \ge 1$. Let G be a graph with $\beta(G) = n$, and let T and $\mathcal{D}$ be as stated in the construction. Also, for each $1 \le i \le \beta(G)$, let $b_ic_id_i$ denote the element of $\mathcal{G}_{c_i}$ containing the edge b_ic_i.

Let $G' = G - b_{\beta(G)}c_{\beta(G)}$, so $\beta(G') = n-1$, and T is a spanning tree for G'. For brevity, put $k = \beta(G) = \beta(G') + 1$. Notice that if $x \in V(G) - \{b_k, c_k\}$, then $\mathcal{G}_x$ is a standard P_2 basis (with common edge in T) for both $\mathcal{P}(G, x)$ and $\mathcal{P}(G', x)$ because $\mathcal{P}(G, x) = \mathcal{P}(G', x)$. Also $\mathcal{G}_{b_k} - \{a_k b_k c_k\}$ is a standard P_2 basis for $\mathcal{P}(G', b_k)$ with common edge in T. (Reason: $\mathcal{G}_{b_k}$ is a standard P_2 basis for $\mathcal{P}(G, b_k)$ with common edge $a_k b_k$ in T, so $a_k b_k c_k$ is the only element of $\mathcal{G}_{b_k}$ that contains the edge $b_k c_k$. All other members of $\mathcal{G}_{b_k}$ are P_2's in G'. Since $\dim(\mathcal{P}(G', b_k)) = \deg_{G'}(b_k) - 1 = \deg_G(b_k) - 2 = \dim(\mathcal{P}(G, b_k)) - 1$, it follows that removing the single path $a_k b_k c_k$ from $\mathcal{G}_{b_k}$ must leave a basis for $\mathcal{P}(G', b_k)$.) For the same reason $\mathcal{G}_{c_k} - \{b_k c_k d_k\}$ is a standard P_2 basis for $\mathcal{P}(G', c_k)$ with common edge in T. Put $\mathcal{G}'_x = \mathcal{G}_x$ if $x \in V(G) - \{b_k, c_k\}$, and $\mathcal{G}'_{b_k} = \mathcal{G}_{b_k} - \{a_k b_k c_k\}$, and $\mathcal{G}'_{c_k} = \mathcal{G}_{c_k} - \{b_k c_k d_k\}$. Then for each $x \in V(G')$, the set $\mathcal{G}'_x$ is a standard P_2 basis for $\mathcal{P}(G', x)$ with common edge in T.

Observe that G' meets the conditions for Construction 3.2 because T is a spanning tree of G', and $E(G') - E(T) = \{b_1 c_1,\ b_2 c_2,\ \ldots,\ b_{\beta(G')} c_{\beta(G')}\}$, and for each $x \in V(G')$ the set $\mathcal{G}'_x$ is a standard P_2 basis for $\mathcal{P}(G', x)$ whose common edge belongs to T, and for each $1 \leq i \leq \beta(G')$, the path $a_i b_i c_i$ is the element of $\mathcal{G}'_{b_i}$ containing $b_i c_i$. By induction, the set $\mathcal{D}' = \{Q[(i-1)i(i+1)] : Q \in \mathcal{G}'{}_x, x \in V(G'), i \in V(C_q)\} - \{[a_i b_i c_i][012] : 1 \leq i \leq k-1\}$ is linearly independent.

Now, $\mathcal{D}$ is a disjoint union $\mathcal{D} = \mathcal{D}' \cup \mathcal{D}''$ where $\mathcal{D}'' = \{Q[(i-1)i(i+1)] : Q \in \{a_k b_k c_k,\ b_k c_k d_k\}, i \in V(C_q)\} - \{\,[a_k b_k c_k][012]\,\}$ is a subset of the diamonds in the subgraph $[a_k b_k c_k d_k] \times C_q$. To show that $\mathcal{D}$ is linearly independent, we just need to show that $\mathcal{D}''$ is linearly independent and span$(\mathcal{D}') \cap$ span$(\mathcal{D}'') = \{0\}$. This is perhaps most easily explained graphically. (Note: we cannot necessarily apply Lemma 3.1 here, for it is possible $a_k = d_k$, in which case $a_k b_k c_k d_k$ is not a tree.) Figure 5 shows the diamonds in $\mathcal{D}''$ for the case $q = 9$, and the picture for the general case (with q odd) is similar. Diamonds $[a_k b_k c_k][(i-1)i(i+1)]$ and $[b_k c_k d_k][(i-1)i(i+1)]$ from $\mathcal{D}''$ are petals in a flower-like arrangement, with the diamond $[a_k b_k c_k][012]$ missing (it does not belong to $\mathcal{D}''$). The shaded region covers the edges of form $(b_k, i)(c_k, i \pm 1)$ which form the set $E(G \times C_q) - E(G' \times C_q)$.

The set $\mathcal{D}''$ is linearly independent as follows. Suppose a sum of diamonds in $\mathcal{D}''$ equals zero. Then the diamond labeled X cannot occur in the sum, because the sum contains no diamond that can cancel the edge $(c_k, 1)(b_k, 2)$ of X. Consequently the diamond labeled Y cannot occur in the sum because the sum has no term to cancel the edge $(b_k, 2)(c_k, 3)$ of Y. Similarly, the diamond Z cannot occur in the sum because the sum has no term to cancel the edge $(c_k, 3)(b_k, 4)$. Continuing around the flower in this pattern we see that the sum has no nonzero terms, so $\mathcal{D}''$ is linearly independent.

Next we show span$(\mathcal{D}') \cap$ span$(\mathcal{D}'') = \{0\}$. Suppose $W \in$ span$(\mathcal{D}') \cap$ span$(\mathcal{D}'')$. Now, the diamonds in $\mathcal{D}'$ are subgraphs of $E(G' \times C_q)$, so none of them have edges in $E(G \times C_q) - E(G' \times C_q)$ (shaded in the figure). Thus W has no edges of this form. At the same time, since W is in span$(\mathcal{D}'')$ it must be a sum of diamonds in $\mathcal{D}''$. Then the diamond X cannot be in the sum because the sum contains no diamond that can cancel the edge $(c_k, 1)(b_k, 2)$ of X. Repeating the argument in the previous paragraph, the sum has no nonzero terms, so $W = 0$.

This completes the proof that $\mathcal{D}$ is linearly independent. To prove the statement about

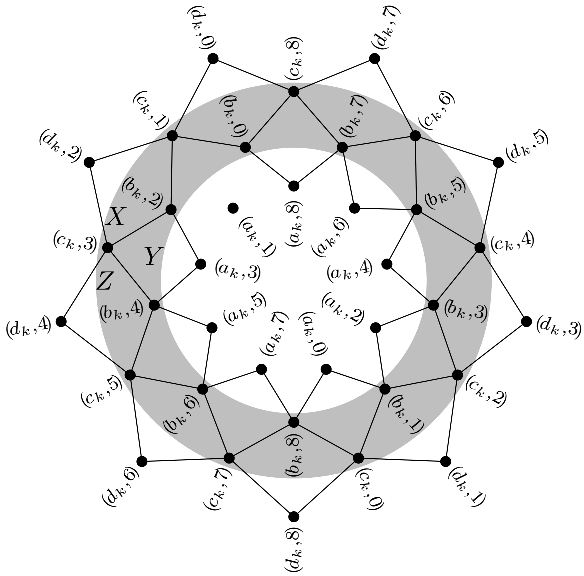


Figure 5: The set $\mathcal{D}''$

the cardinality, note that the definition of $\mathcal{D}$ yields

$$\begin{aligned}
|\mathcal{D}| &= \left|\{Q[(i-1)i(i+1)] : Q \in \mathcal{G}_x, x \in V(G), i \in V(C_q)\}\right| - \\
&\qquad \left|\{[a_ib_ic_i][012] : 1 \le i \le \beta(G)\}\right| \\
&= \sum_{x\in V(G)} |\mathcal{G}_x| q \;-\; \beta(G) \\
&= q\left(\sum_{x\in V(G)} \deg_G(x) - 1\right) - (|E(G)| - |V(G)| + 1) \\
&= q(2|E(G)| - |V(G)|) - (|E(G)| - |V(G)| + 1) \\
&= (2|E(G)|q - |V(G)|q + 1) - (|E(G)| - |V(G)| + 1) - 1 \\
&= \beta(G \times C_q) - \beta(G) - 1.
\end{aligned}$$

The proof is now complete. □

Before moving on, we visit a consequence of Construction 3.2 that will be useful.

Lemma 3.3. *Suppose G is a connected non-bipartite graph, e is an edge of C_q and P is the path $C_q - e$. Let $ab \in E(P)$, and regard $G \times ab \cong G \times K_2$ as a subgraph of $G \times P$. Then $\mathcal{C}(G \times P) = \mathcal{C}(G \times ab) \bigoplus \mathcal{D}(G \times P)$. Moreover, if $C = A + B$ with $A \in \mathcal{C}(G \times ab)$ and $B \in \mathcal{D}(G \times P)$, then $|C| \ge |A|$.*

Proof. Let the partite sets of P be X_a and X_b, with $a \in X_a$ and $b \in X_b$. Observe that there is a homomorphism $\rho : G \times P \to G \times ab$ defined as $\rho(x, i) = (x, a)$ if $i \in X_a$

and $\rho(x,i) = (x,b)$ if $i \in X_b$. Note that ρ is the identity on $G \times ab$, and it thus induces a projection $\rho^* : \mathcal{C}(G \times P) \to \mathcal{C}(G \times ab)$. Therefore $\mathcal{C}(G \times P) = \mathcal{C}(G \times ab) \bigoplus \ker(\rho^*)$, so $\dim(\ker(\rho^*)) = \beta(G \times P) - \beta(G \times ab)$. Now $\rho^*(D) = 0$ for any diamond D, so $\mathcal{D}(G \times P) \subseteq \ker(\rho^*)$. We will finish the proof by producing a linearly independent set of diamonds in $\mathcal{D}(G \times P)$ of cardinality $\beta(G \times P) - \beta(G \times ab)$.

We can create an independent set of diamonds in $\mathcal{D}(G \times P)$ by removing from the set $\mathcal{D}$ of Construction 3.2 all diamonds that are not in $G \times P$. If $e = c(c+1)$, we remove the diamonds of form $Q[(c-1)c(c+1)]$ and $Q[c(c+1)(c+2)]$ and obtain the independent set $\mathcal{B} = \{Q[(i-1)i(i+1)] : Q \in \mathcal{G}_x, x \in V(G), i \in \mathbb{Z}_q - \{c, c+1\}\}$ of diamonds in $G \times P$. Observe

$$
\begin{aligned}
|\mathcal{B}| &= \sum_{x \in V(G)} |\mathcal{G}_x|(q-2) = (q-2) \sum_{x \in V(G)} (deg_G(x) - 1) \\
&= (q-2)(2|E(G)| - |V(G)|) \\
&= \big(2|E(G)|(q-1) - V(G)|q + 1\big) - \big(2|E(G)| - 2|V(G)| + 1\big) \\
&= \beta(G \times P) - \beta(G \times ab).
\end{aligned}
$$

Finally, suppose $C = A + B$ as in the statement of the lemma. Now, $|C| \geq |\rho^*(C)|$ because any edge in $\rho^*(C)$ must be the image under ρ^* of at least one edge in C. Since $\rho^*(C) = \rho^*(A+B) = \rho^*(A) + \rho^*(B) = \rho^*(A) = A$, we have $|C| \geq |A|$. □

4 The Product of Two Odd Cycles

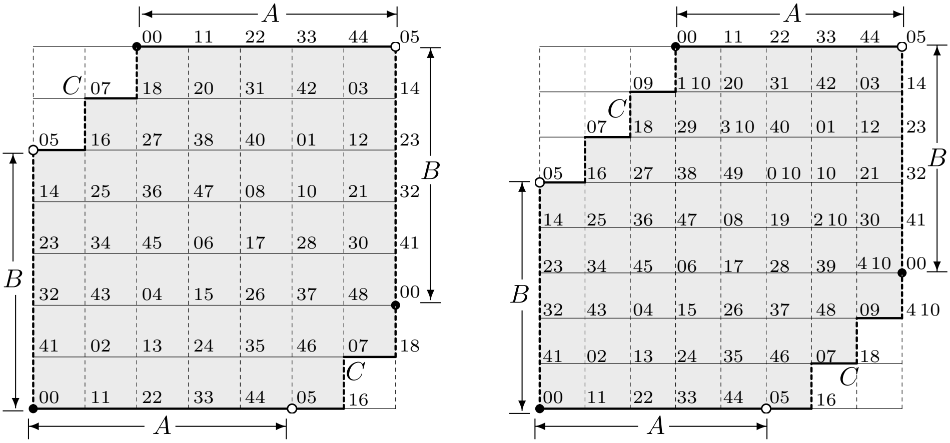
In this section we take up the problem of finding an MCB for the product of cycles $C_p \times C_q$ where p and q are odd and at least one is greater than 3. For simplicity assume $p \leq q$. This is a special case of our ultimate problem of finding an MCB of $G \times C_q$, but treating it now will help us understand some of the subtleties of the general problem and put us in a position to better understand and motivate our general construction. The discussion is informal and is intended for illumination only, and any lack of rigor will be compensated in the subsequent section. Though the arguments used in this section are topological, those that follow will be entirely combinatorial.

Observe that $C_p \times C_q$ can be embedded on the torus with pq square regions whose boundaries are the diamonds of $C_p \times C_q$. This is illustrated for $C_5 \times C_9$ in Figure 6(a) and for $C_5 \times C_{11}$ in Figure 6(b). In each case the torus is an identification space obtained by identifying paths A (of length p), paths B (of length p), and the zig-zag path C (of length $q-p$). The general case is illustrated in figures 7(a) and 7(b).

The set of all pq diamonds is linearly dependent, for if they are all added together their edges will cancel pair-by-pair. But set $\mathcal{D}$ of Construction 3.2 is independent and $|\mathcal{D}| = \beta(C_p \times C_q) - \beta(C_q) - 1 = (2pq - pq + 1) - 1 - 1 = pq - 1$. Thus $\mathcal{D}$ contains all but one diamond, and the missing one is the sum everything in $\mathcal{D}$, so $\mathcal{D}$ is a basis for $\mathcal{D}(C_p \times C_q)$.

Let's now use the Greedy Algorithm to obtain an MCB. There are no cycles of length less than 4, so begin by setting $\mathcal{M} := \mathcal{D}$. Since $\beta(C_p \times C_q) = pq + 1 = |\mathcal{M}| + 2$, there are just two more cycles to append.

As an aid in finding these two cycles, we claim any even cycle $Z \in \mathcal{C}(C_p \times C_q)$ with $|Z| < 2p$ is a sum of diamonds, and is thus already in span$(\mathcal{M})$. For if Z is such an even cycle, the homomorphism $\pi^*_{C_p} : \mathcal{C}(C_p \times C_q) \to \mathcal{C}(C_p) = \{0, C_p\}$ must send Z to an even cycle, so $\pi^*_{C_p}(Z) = 0$. Then for any edge $e = ab \in E(C_p)$, cycle Z must have



$C_5 \times C_9$ (a) $C_5 \times C_{11}$ (b)

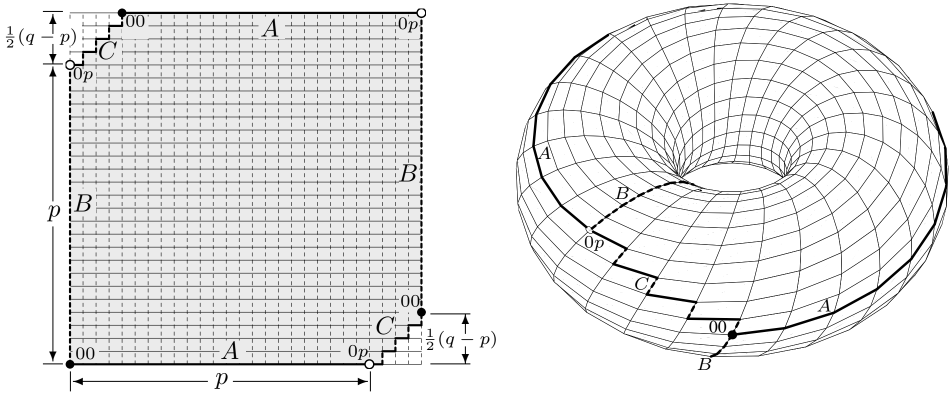
Figure 6: The graph $C_p \times C_q$ on the torus

an even number m_e of edges of form $(a, x)(b, y)$ for which $\pi_{C_p}((a, x)(b, y)) = ab$. Since $2p > |Z| = \sum_{e \in E(C_p)} m_e$, it follows that $m_e = 0$ for some $e \in E(C_p)$. Hence Z is a cycle in the graph $(C_p - e) \times C_q$. By applying the same argument to the factor C_q (and using $p \leq q$) we see C_q must have some edge f for which Z is a cycle in the product $(C_p - e) \times (C_q - f)$ of paths. A product of paths has two planar components which can be embedded in the plane so that the boundaries of all interior regions are diamonds. By MacLane's theorem, these diamonds span the cycle space, so Z is a sum of diamonds.

In Figure 6 the edges of the products are colored solid and dashed according to whether they run horizontally or vertically in the grids. With this coloring, every diamond has two edges of each color, so any element of span($\mathcal{M}$) has an even number of edges of each color. Now the even cycle $A + B$ of length $2p$ has p (odd) edges of each color, $A + B \notin$ span($\mathcal{M}$). Further, $A + C$ and $B + C$ are cycles of length q (odd), so they are certainly not in span($\mathcal{M}$). Since $A + B = (A + C) + (B + C)$, it follows that appending to $\mathcal{M}$ any two elements of $\{A + B, A + C, B + C\}$ will produce a basis.

Now continue with the Greedy Algorithm. We know that if an even cycle is appended to $\mathcal{M}$, the even cycle must have length no less than $2p$. At the same time, since $C_p \times C_q$ has odd cycles, at least one odd cycle must appear in an MCB, and such an odd cycle can have length no less than q. Therefore if $q < 2p$, then $\mathcal{M}$ can be extended to an MCB by appending to it the odd cycles $A + C$ and $B + C$ of length q. On the other hand, if $2p < q$, then $\mathcal{M}$ is extended to an MCB by appending to it the even cycle $A + B$ of length $2p$ the odd cycle $A + C$ of length q. This proves the following result.

Proposition 4.1. *Suppose p and q are odd integers, $p \leq q$ and $\max\{p, q\} > 3$.*
If $q < 2p$, then $C_p \times C_q$ has an MCB consisting of $pq - 1$ squares and two q-cycles.
If $2p < q$, then $C_p \times C_q$ has an MCB consisting of $pq - 1$ squares, a $2p$-cycle and a q-cycle.



(a) (b)

Figure 7: The graph $C_p \times C_q$ on the torus

Figures 6(a) and 6(b) illustrate this proposition. In Figure 6(a) $C_5 \times C_9$ has an MCB consisting of 44 diamonds, and two 9-cycles $A + C$ and $B + C$. In Figure 6(a) $C_5 \times C_{11}$ has an MCB consisting of 54 diamonds, one 10-cycle $A + B$ and one 11-cycle $A + C$.

5 An MCB for $G \times C_q$

In the previous section we constructed an MCB for $C_p \times C_q$ where p, q are odd, $p \leq q$ and $\max\{p, q\} > 3$. We now generalize this by replacing the factor C_p with a connected graph G whose shortest odd cycle has length p. That is, we construct an MCB for $G \times C_q$ where $p \leq q$ and $\max\{p, q\} > 3$. Under these hypotheses every odd cycle in $G \times C_q$ has length at least q, so $G \times C_q$ is triangle-free.

As was the case for $G \times K_2$ in Section 2, we should not expect an MCB of $G \times C_q$ to correspond in any way to an MCB of G. And as in Section 2 our approach here will be to replace G with its symmetric digraph $\overleftrightarrow{G}$ and transfer minimal cycle structures of $\overleftrightarrow{G}$ to $G \times C_q$. However, since $G \times C_q$ has odd cycles (unlike $G \times K_2$), the anti-cycle space $\mathcal{A}(\overleftrightarrow{G})$ (whose elements all possess an even number of arcs) does not have an adequate cycle structure for the job at hand, and we will have to enlarge it slightly. The relevant definitions follow.

As in Section 2, let $\overleftrightarrow{G}$ denote the symmetric digraph on G with arc set $E(\overleftrightarrow{G}) = \{\overrightarrow{xy}, \overrightarrow{yx} : xy \in E(G)\}$. A pair $\{\overrightarrow{xy}, \overrightarrow{yx}\}$ is called a **double edge** of $\overleftrightarrow{G}$. A sub-digraph of $\overleftrightarrow{G}$ is **symmetric** if whenever $\overrightarrow{xy}$ is one of its arcs, then $\overrightarrow{yx}$ is also one of its arcs (i.e. if every one of its arcs is part of a double edge.)

Let $\mathcal{C}(\overleftrightarrow{G})$ be the kernel of the linear map $\lambda : \mathcal{E}(\overleftrightarrow{G}) \to \mathcal{V}(G)$ defined as $\lambda(\overrightarrow{xy}) = \{x\} + \{y\}$. One easily checks that $\mathcal{C}(\overleftrightarrow{G})$ consists of the edge sets of the sub-digraphs of $\overleftrightarrow{G}$ for which at each vertex the sum of the in- and out-degree is even. Observe that $\mathcal{A}(\overleftrightarrow{G})$ is a proper subspace of $\mathcal{C}(\overleftrightarrow{G})$, provided that G has at least one edge. Indeed, any double edge $\{\overrightarrow{xy}, \overrightarrow{yx}\} \in \mathcal{E}(\overleftrightarrow{G})$ is in $\mathcal{C}(\overleftrightarrow{G})$ but not in $\mathcal{A}(\overleftrightarrow{G})$. The range of λ is the subspace of $\{X \subset V(G) : |X| \text{ is even}\}$ of $\mathcal{V}(G)$, and its dimension is $|V(G)| - 1$. (Because it is the kernel of the surjective linear map $\mathcal{V}(G) \to \mathbb{F}_2$ defined as $X \to |X|$ (mod 2).) Therefore, since $\operatorname{rank}(\lambda) + \dim(\ker(\lambda)) = \dim(\mathcal{E}(\overleftrightarrow{G})) = 2|E(G)|$, we see $\dim(\mathcal{C}(\overleftrightarrow{G})) = 2|E(G)| - |V(G)| + 1$.

Just as we regard elements of $\mathcal{C}(G)$ as (eulerian) subgraphs of G, we regard elements of $\mathcal{C}(\overleftrightarrow{G})$ as the sub-digraphs of $\overleftrightarrow{G}$ for which the total degree (in-degree plus out-degree) of each vertex is even. Recall that an **orientation** of a graph is an assignment of a direction to each of its edges. Thus if $A \in \mathcal{C}(G)$, any orientation of A is in $\mathcal{C}(\overleftrightarrow{G})$. However an orientation of A has no double edges, while an element of $\mathcal{C}(\overleftrightarrow{G})$ may have double edges. Consequently though $\mathcal{C}(\overleftrightarrow{G})$ contains all the orientations of eulerian subgraphs of G, not every element of $\mathcal{C}(\overleftrightarrow{G})$ is such an orientation.

We now define a projection $\pi : \mathcal{C}(G \times C_q) \to \mathcal{C}(\overleftrightarrow{G})$. Consider the linear map $\pi : \mathcal{E}(G \times C_q) \to \mathcal{E}(\overleftrightarrow{G})$ which acts on the basis $E(G \times C_q)$ as

$$\pi((v,i)(w,j)) = \begin{cases} \overrightarrow{vw} & \text{if } j = i+1 \\ \overrightarrow{wv} & \text{if } j = i-1. \end{cases}$$

It is straightforward to check that this restricts to a linear map $\pi : \mathcal{C}(G \times C_q) \to \mathcal{C}(\overleftrightarrow{G})$.

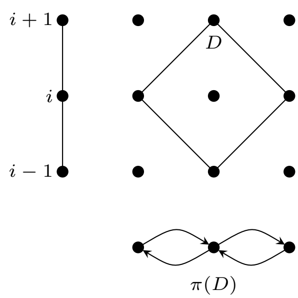


Figure 8: Projection of a diamond

As an example, observe that (as illustrated in Figure 8) if D is a diamond, then $\pi(D)$ consists of two double edges. It follows that if $Y \in \mathcal{D}(G \times C_q)$, then $\pi(Y)$ consists of an even number of double edges.

Our reason for constructing $\mathcal{C}(\overleftrightarrow{G})$ is to have a richer version of $\mathcal{C}(G)$ so that, roughly, we may ultimately be able to lift some minimal cycle structure in $\mathcal{C}(\overleftrightarrow{G})$ to $\mathcal{C}(G \times C_q)$. We've seen that if $Y \in \mathcal{D}(G \times C_q)$, then $\pi(Y)$ is a symmetric sub-digraph of $\overleftrightarrow{G}$ that consists of an even number of double edges. By Construction 3.2 we already have a set $\mathcal{D}$ of diamonds that spans $\mathcal{D}(G \times C_q)$, so we are not interested in lifting such sub-digraphs Y to cycles in $\mathcal{D}(G \times C_q)$. Thus we next cut down the size of $\mathcal{C}(\overleftrightarrow{G})$ by forming the quotient of if with the space of symmetric digraphs with an even number of double edges.

Let $\mathcal{V} = \{Y \in \mathcal{C}(\overleftrightarrow{G}) : Y \text{ is symmetric and } |Y| \equiv 0 \text{ (mod 4)}\}$. Note $\mathcal{V}$ is precisely the subset of $\mathcal{C}(\overleftrightarrow{G})$ whose elements are symmetric digraphs with an even number of double edges. (Each double edge contains two opposing arcs, so an even number of double edges yields a total number of arcs that is a multiple of 4.) Clearly $\mathcal{V}$ is a subspace of $\mathcal{C}(\overleftrightarrow{G})$. If we identify the double edges in $\overleftrightarrow{G}$ with edges in G, then $\mathcal{V}$ is identified with the subspace $\mathcal{W} = \{X \subseteq E(G) : |X| \text{ is even }\}$ of $\mathcal{E}(G)$. Since $\dim(\mathcal{W}) = |E(G)| - 1$, we obtain $\dim(\mathcal{V}) = |E(G)| - 1$.

Now consider the quotient $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$. The dimension of this space is $\dim(\mathcal{C}(\overleftrightarrow{G})) - \dim(\mathcal{V}) = (2|E(G)| - |V(G)| + 1) - (|E(G)| - 1) = \beta(G) + 1$. Since its dimension is just one more than the dimension of $\mathcal{C}(G)$, we would expect the structure of $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$ to be similar to — but slightly richer than — the structure of $\mathcal{C}(G)$. In fact we will soon see that elements of $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$ can be lifted to part of an MCB for $G \times C_q$, whereas that is not necessarily possible for lifts of elements of $\mathcal{C}(G)$.

Example 5.1. As a specific example of $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$, consider the case $G = C_p$, where p is odd. Note that $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$ has dimension $\beta(G)+1 = 2$. Let A_1 be the digraph obtained from C_p by giving it the orientation where each arc is directed from i to $i+1$. Let A_2 be A_1 with arcs reversed, that is the arcs in A_2 are directed from i to $i-1$. Since neither A_1 nor A_2 is symmetric, $A_1 + \mathcal{V}$ and $A_2 + \mathcal{V}$ are nonzero in $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$. Also $A_1 + A_2$ is a symmetric digraph with p (odd) double edges, so $A_1 + A_2 \notin \mathcal{V}$ It follows that the set $\{A_1{+}\mathcal{V},\ A_2{+}\mathcal{V}\}$ is linearly independent in $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$. Thus $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V} = \{\mathcal{V},\ A_1{+}\mathcal{V},\ A_2{+}\mathcal{V},\ (A_1{+}A_2){+}\mathcal{V}\}$.

Definition 5.2. Let $\overrightarrow{C}_n$ be the digraph with vertices $\mathbb{Z}_n$ and with arcs directed from i to $i+1$ for each $i \in \mathbb{Z}_n$. A sub-digraph C in $\overleftrightarrow{G}$ is called a **directed cycle** if it is isomorphic to $\overrightarrow{C}_n$.

In the next lemma we need the linear function $f : \mathcal{E}(\overleftrightarrow{G}) \to \mathcal{E}(G)$ defined on basis $E(\overleftrightarrow{G})$ as $f(\overrightarrow{xy}) = xy$. Thus f simply eliminates all double edges of its argument and "forgets" the orientation of the remaining arcs. The kernel of f is the space of symmetric digraphs in $\mathcal{E}(\overleftrightarrow{G})$. It is easy to check that f restricts to a map $f : \mathcal{C}(\overleftrightarrow{G}) \to \mathcal{C}(G)$.

Lemma 5.3. *If G is non-bipartite, then $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$ is spanned by the elements $A + \mathcal{V}$ where A is a directed odd cycle or an anti-cycle.*

Proof. Let $\mathcal{A} = \{A_1, A_2, \ldots, A_{\beta(G)}\}$ be a basis of $\mathcal{C}(G)$ consisting of simple cycles (an MCB will suffice). For each index i, give A_i an orientation that makes it an anti-cycle if $|A_i|$ is even, or a directed cycle if $|A_i|$ is odd. Call the resulting digraph A_i'. Thus we have $f(A_i') = A_i$. The set $\mathcal{A}' = \{A_1' + \mathcal{V},\ A_2' + \mathcal{V}, \ldots, A_{\beta(G)}' + \mathcal{V}\}$ is linearly independent in $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$ for the following reason. Suppose $\sum_{i\in I}(A_i' + \mathcal{V}) = \mathcal{V}$, where $I \subseteq \{1, 2, \ldots, \beta(G)\}$. Then $\sum_{i\in I} A_i' \in \mathcal{V}$. Since any element of $\mathcal{V}$ is symmetric we have $f\left(\sum_{i\in I} A_i'\right) = \sum_{i\in I} A_i = 0$. Thus $I = \emptyset$, showing $\mathcal{A}'$ is independent.

Now let $A_{\beta(G)+1}'$ be the symmetric digraph on a simple odd cycle of G. (That is it is obtained by replacing each edge of a simple odd cycle of G with a double edge.) Then $A_{\beta(G)+1}'$ is an anti-cycle. Now even though $A_{\beta(G)+1}'$ is symmetric, it has an odd number of double edges so $A_{\beta(G)+1}' + \mathcal{V}$ is nonzero in $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$. We now show that it is not a linear combination of elements in $\mathcal{A}'$. Suppose to the contrary that $A_{\beta(G)+1}' + \mathcal{V} = \sum_{i\in I}(A_i' + \mathcal{V})$. Then $A_{\beta(G)+1}' + \mathcal{V} = \left(\sum_{i\in I} A_i'\right) + \mathcal{V}$, so $\sum_{i\in I} A_i'$ is symmetric, as it is the sum of the symmetric digraph $A'_{\beta(G)+1}$ and a symmetric graph in $\mathcal{V}$. Applying f we have $\sum_{i\in I} A_i = 0$, a contradiction.

Thus we have linearly independent set $\{A'_1+\mathcal{V},\ A'_2+\mathcal{V},\ \ldots,\ A'_{\beta(G)}+\mathcal{V},\ A'_{\beta(G)+1}+\mathcal{V}\}$ with each A'_i is an anticycle or a directed odd cycle. Since $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$ is known to have dimension $\beta(G)+1$, we are done. □

If $\widetilde{A} \in \mathcal{C}(G \times C_q)$ and $\pi(\widetilde{A}) = A$, then certainly $|\widetilde{A}| \geq |A|$ because each arc in A is the projection of at least one edge in $\widetilde{A}$. Moreover $|A|$ is odd if and only if $|\widetilde{A}|$ is odd, and in such cases $|\widetilde{A}| \geq \max\{q, |A|\}$ because $G \times C_q$ has no odd cycles of length less than q. The next lemmas show that if A is an anti-cycle or a directed odd cycle, then there is some $\widetilde{A} \in \mathcal{C}(G \times C_q)$ for which $\pi(\widetilde{A}) = A$ (modulo $\mathcal{V}$), and for which $\widetilde{A}$ attains the minimum length $|A|$ if A is an anti-cycle, or $\max\{q, |A|\}$ if A is a directed odd cycle. Let $\pi' : \mathcal{C}(G \times C_q) \to \mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$ be the map $\pi'(C) = \pi(C)+\mathcal{V}$.

Lemma 5.4. *If A is an anti-cycle in $\mathcal{C}(\overleftrightarrow{G})$, then there is a cycle $\widetilde{A} \in \mathcal{C}(G \times C_q)$ with $|\widetilde{A}| = |A|$ and $\pi(\widetilde{A}) = A$. In particular $\pi'(\widetilde{A}) = A+\mathcal{V}$.*

Proof. Given anti-cycle A, let $\widetilde{A} = \{(x, 0)(y, 1) : \overrightarrow{xy} \in E(A)\}$, as illustrated in Figure 3. By construction $|\widetilde{A}| = |A|$ and $\pi(\widetilde{A}) = A$. □

Lemma 5.5. *If A is a directed odd cycle in $\mathcal{C}(\overleftrightarrow{G})$, then there is a cycle $\widetilde{A} \in \mathcal{C}(G \times C_q)$ with $|\widetilde{A}| = \max\{q, |A|\}$ and $\pi'(\widetilde{A}) = A+\mathcal{V}$.*

Proof. Say A has n vertices. Label its vertices with the elements of $\mathbb{Z}_n$ so that each arc of A has form $\overrightarrow{i(i+1)}$. We consider four cases.

Case (a). Suppose $q > n$ and $q - n \equiv 0 \pmod 4$. Let $\widetilde{A}$ be the concatenation of paths

$$\begin{aligned} L &= (0,0)(1,1)(2,2)\ldots(n-1,n-1)(0,n) \\ \text{and} \quad M &= (0,n)(1,n+1)(0,n+2)(1,n+3)\ldots(0,0) \end{aligned}$$

of lengths n and $q-n$ respectively, which are shown solid and dashed in Figure 9(a). Then $|\widetilde{A}| = q = \max\{q, |A|\}$. Further, $\pi(L) = A$. Moreover, π sends every two successive edges of M to $\overrightarrow{01} + \overrightarrow{10}$. Since the length of M is a multiple of 4, it follows that $\pi(M) = 0$. Therefore $\pi(\widetilde{A}) = \pi(L) + \pi(M) = A + 0 = A$, so $\pi'(\widetilde{A}) = A+\mathcal{V}$, as required.

Case (b). Suppose $q > n$ and $q - n \equiv 2 \pmod 4$. If we used L and M from the previous case, then $\pi(L) = \overrightarrow{01} + \overrightarrow{10}$, so $\pi(\widetilde{A})$ would be A with the arc 01 replaced with 10. Instead, let $\widetilde{A}$ be the concatenation of

$$\begin{aligned} L &= (0,0)(n-1,1)(n-2,2)\ldots(1,n-1)(0,n) \\ \text{and} \quad M &= (0,n)(1,n+1)(0,n+2)(1,n+3)\ldots(0,0) \end{aligned}$$

which are shown solid and dashed, respectively in Figure 9(b). Then $\pi(L)$ is the reverse orientation of A, so $\pi(L) + A$ is a symmetric graph with n (odd) double edges. Also $\pi(M) = \overrightarrow{01} + \overrightarrow{10}$, so $\pi(L) + \pi(M) + A = \pi(\widetilde{A}) + A$ is a symmetric graph with $n - 1$ (even) double edges. Hence $\pi(\widetilde{A}) + A \in \mathcal{V}$, so $\pi'(\widetilde{A}) = A+\mathcal{V}$, as required.

Case (c). Suppose $q \leq n$ and $n-q \equiv 0 \pmod 4$. Let $\widetilde{A}$ be the concatenation of paths

$$\begin{aligned} L &= (0,0)(1,1)(2,2)(3,3)\dots(q,0) \\ \text{and} \quad M &= (q,0)(q+1,1)(q+2,0)(q+3,1)(q+4,0)\dots(n-1,1)(0,0) \end{aligned}$$

of lengths q and $n-q$, which are shown solid and dashed, respectively in Figure 9(c). Notice that $|\widetilde{A}| = |A| = \max\{q, |A|\}$. Also $\pi(\widetilde{A}) = \pi(L) + \pi(M)$ is the directed graph obtained from A by reversing every other arc in the directed path $q(q+1)(q+2)\dots 0$ in A. Since the number of arcs in this path is a multiple of 4, $\pi(\widetilde{A})$ is just A with an even number of arcs reversed. It follows that $\pi(\widetilde{A}) + A$ is a symmetric graph with an even number of double edges, so $\pi(\widetilde{A}) + A \in \mathcal{V}$, hence $\pi'(\widetilde{A}) = A + \mathcal{V}$.

Case (d). Suppose $q \leq n$ and $n-q \equiv 2 \pmod 4$. Let $\widetilde{A}$ be the concatenation of paths

$$\begin{aligned} L &= (0,0)(n-1,1)(n-2,2)(n-3,3)\dots(n-q,0) \quad \text{and} \\ M &= (n-q,0)(n-q+1,1)(n-q+2,0)(q+3,1)(n-q+4,0)\dots(n-1,1)(0,0) \end{aligned}$$

of lengths q and $n-q$ and reason as above. □

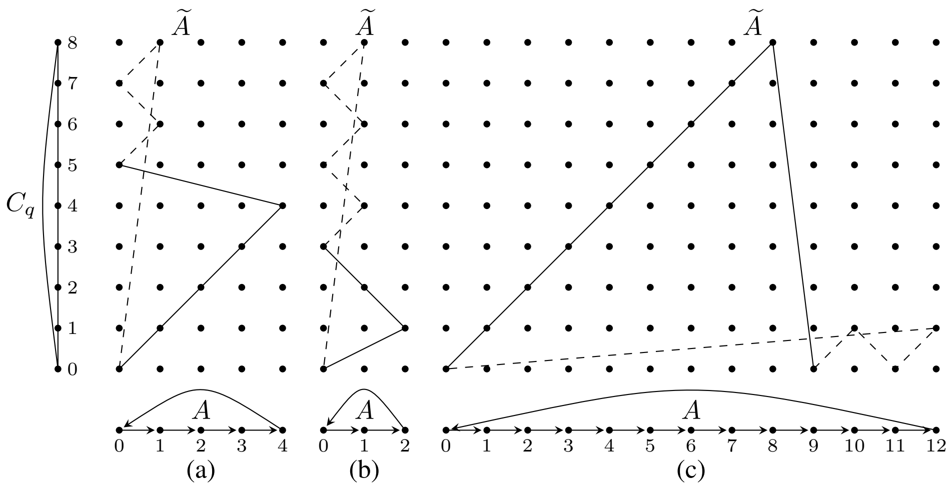


Figure 9: Lifts of directed odd cycles

In order to lift a minimum cycle structure of $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$ to part of an MCB of $G \times C_q$ it will be necessary to weight elements $A + \mathcal{V}$ not by $|A|$, but by the number of edges in a lift of A. Hence the following definition, motivated by the previous two lemmas.

Definition 5.6. If $A \in \mathcal{C}(\overleftrightarrow{G})$ is an anti-cycle or a directed odd cycle then its q**-weight** is the integer

$$w_q(A) = \begin{cases} |A| & \text{if } A \text{ is an anti-cycle} \\ \max\{q, |A|\} & \text{if } A \text{ is a directed odd cycle.} \end{cases}$$

A basis $\mathcal{A} = \{A_1 + \mathcal{V},\ A_2 + \mathcal{V}, \dots, A_{\beta(G)+1} + \mathcal{V}\}$ for $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$ is called a **minimum** q**-weight basis** if each A_i is an anti-cycle or a directed odd cycle and the total q-weight

$\sum_{i=1}^{\beta(G)+1} w_q(A_i)$ has the minimum possible value among all such bases. (A minimum q-weight basis exists by Lemma 5.3.)

We can finally state our construction for an MCB of $G \times C_q$.

Construction 5.7. *(An MCB for $G \times C_q$ where G is connected and non-bipartite, q is odd, the shortest odd cycle in G has length $p \leq q$, and $\max\{p, q\} > 3$.)*

1. *Let $\mathcal{D}$ be the set of diamonds from Construction 3.2.*
2. *Let $\mathcal{A} = \{A_1 + \mathcal{V}, A_2 + \mathcal{V}, \ldots, A_{\beta(G)+1} + \mathcal{V}\}$ be a minimum q-weight basis for $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$.*
3. *Let $\widetilde{\mathcal{A}} = \{\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_{\beta(G)+1}\} \subseteq \mathcal{C}(G \times C_q)$ be such that $\pi(\widetilde{A}_i) + \mathcal{V} = A_i + \mathcal{V}$ and $|\widetilde{A}_i| = w_q(A_i)$ for each index $1 \leq i \leq \beta(G) + 1$. (As in Lemmas 5.4 and 5.5.)*

Then *$\mathcal{B} = \widetilde{\mathcal{A}} \cup \mathcal{D}$ is an MCB for $G \times C_q$.*

Before proving this, let's look at a simple example.

Example 5.8. Consider the case of constructing an MCB of $G \times C_q$ where $G = C_p$, which was addressed in Section 4. Assume, as we did in that section, that $p \leq q$ and $\max\{p, q\} > 3$. In Step 1 of Construction 5.7 the set $\mathcal{D}$ is formed, and it has $\beta(C_p \times C_q) - \beta(C_p) - 1 = pq - 1$ diamonds.

Now we move on to Step 2. We computed $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V}$ in Example 5.1. Recall that $\mathcal{C}(\overleftrightarrow{G})/\mathcal{V} = \{\mathcal{V},\ A_1 + \mathcal{V},\ A_2 + \mathcal{V},\ A_3 + \mathcal{V}\}$. where A_1 and A_2 are opposite orientations on C_p, and $A_3 = A_1 + A_2$ is an anti-cycle of length $2p$. Note that $w_q(A_1) = w_q(A_2) = q$, and $w_q(A_3) = 2p$. Depending on whether $q < 2p$ or $2p < q$, we would choose as our minimum q-weight basis either $\mathcal{A} = \{A_1 + \mathcal{V},\ A_2 + \mathcal{V}\}$ or $\mathcal{A} = \{A_1 + \mathcal{V},\ A_3 + \mathcal{V}\}$. In the first case, $\widetilde{\mathcal{A}} = \{\widetilde{A}_1, \widetilde{A}_2\}$ consists of two q-cycles. In the second case $\widetilde{\mathcal{A}} = \{\widetilde{A}_1, \widetilde{A}_3\}$ consists of one q-cycle and one $2p$-cycle. Notice that the resulting MCB $\widetilde{\mathcal{A}} \cup \mathcal{D}$ agrees with Proposition 4.1.

Now we prove that our construction is valid.

Proof. First we confirm that $\mathcal{B}$ is a basis of $\mathcal{C}(G \times C_q)$. Note that $|\mathcal{B}| = |\widetilde{\mathcal{A}} \cup \mathcal{D}| = \big(\beta(G) + 1\big) + \big(\beta(G \times C_q) - \beta(G) - 1\big) = \beta(G \times C_q)$, so we just need to show that $\mathcal{B}$ is independent. Index $\mathcal{D}$ as $\mathcal{D} = \{D_d : 1 \leq d \leq \beta(G \times C_q) - \beta(G) - 1\}$ and suppose

$$\sum_{i \in I} \widetilde{A}_i + \sum_{d \in \Delta} D_d = 0, \tag{5.1}$$

where $I \subseteq \{1, 2, \ldots, \beta(G) + 1\}$ and Δ is a subset of the set that idexes $\mathcal{D}$. We want to show $I = \Delta = \emptyset$. Taking π' of both sides of Equation (5.1) produces $\sum_{i \in I} \pi'(\widetilde{A}_i) = \mathcal{V}$, which by choice of the $\widetilde{A}_i$ becomes $\sum_{i \in I}(A_i + \mathcal{V}) = \mathcal{V}$, so $I = \emptyset$. Thus $\sum_{d \in \Delta} D_d = 0$ by Equation (5.1), but since $\mathcal{D}$ is linearly independent by construction we have $\Delta = \emptyset$. Thus $\mathcal{B}$ is a basis.

To prove that $\mathcal{B}$ is minimal, consider any $C \in \mathcal{C}(G \times C_q)$ and put

$$C = \sum_{i \in I} \widetilde{A}_i + \sum_{d \in \Delta} D_d. \tag{5.2}$$

According to Proposition 1.1, we just need to show that $|C|$ is not smaller than the length of any term in this sum. Certainly $|C| \geq |D_d| = 4$ for each $d \in \Delta$ because the condition $\max\{p, q\} > 3$ implies that $G \times C_q$ has no triangles, and hence no cycles of length smaller than 4. We just need to confirm $|C| \geq |\widetilde{A}_i|$ for each $i \in I$. We will do this in cases.

Case 1. Suppose $|C| \geq q$. Take π' of both sides of (5.2) to obtain

$$\pi(C)+\mathcal{V} = \sum_{i\in I}(A_i+\mathcal{V}). \tag{5.3}$$

Let S be the maximum symmetric sub-digraph of $\pi(C)$. (So $S = 0$ if $\pi(C)$ has no double edges.) Then $\pi(C) = Y + S$ where Y consists of all the arcs of $\pi(C)$ that are not arcs of S. (So $Y = 0$ if $\pi(C)$ is symmetric.) Thus Y has no double edges, so it is an orientation of the eulerian subgraph $f(Y)$ of G. (Recall that f is the function that "erases" the orientation of Y.) Now, $f(Y)$ decomposes into a disjoint union of simple cycles, so $Y = \sum_{j=1}^{n} B_j$ where the B_j are orientations on pairwise disjoint simple cycles of G. For each index j, let B'_j be an orientation on $f(B_j)$ such that B'_j is a simple directed odd cycle if $|B_j|$ is odd, or an anti-cycle if $|B_j|$ is even. Then $B_j + B'_j$ is symmetric for each j, so $S + \sum_{j=1}^{n}(B_j + B'_j)$ is symmetric, though it may or may not be in $\mathcal{V}$. Let B_0 be a shortest odd (simple) cycle in G, and let B'_0 be an orientation of B_0 such that B'_0 is a directed odd cycle. Let B'_{-1} be the orientation of B_0 that is opposite to the orientation of B'_0 (i.e. B'_{-1} is B'_0 with the arcs reversed). Thus $B'_0 + B'_{-1}$ is a symmetric digraph with p (odd) double edges. Observe

$$b(B'_{-1} + B'_0) + S + \sum_{j=1}^{n}(B_j + B'_j) \;\; \in \;\; \mathcal{V},$$

where b is 0 or 1 according to whether the symmetric digraph $S + \sum_{j=1}^{n}(B_j + B'_j)$ has an even or odd number of double edges. Since $\pi(C) = S + Y = S + \sum_{j=1}^{n} B_j$, the above equation tells us

$$\pi(C)+\mathcal{V} = b(B'_{-1}+\mathcal{V}) + b(B'_0+\mathcal{V}) + \sum_{j=1}^{n}(B'_j+\mathcal{V}). \tag{5.4}$$

Recall that each B'_j in this sum is either an anti-cycle or a simple directed odd cycle. Since $|C| \geq q \geq p$ we have $|C| \geq q = \max\{q, |B'_0|\} = w_q(B'_0)$, and similarly $|C| \geq w_q(B'_{-1})$. Also for $1 \leq j \leq n$ we have $|\pi(C)| \geq |B'_j|$ by construction, so $|C| \geq \max\{q, |\pi(C)|\} \geq \max\{q, |B'_j|\} \geq w_q(B_j)$. Therefore

$$|C| \geq w_q(B'_j) \text{ for } -1 \leq j \leq n. \tag{5.5}$$

Now for each $-1 \leq j \leq n$ we have $B'_j+\mathcal{V} = \sum_{i\in J_j}(A_i+\mathcal{V})$ for an appropriate index set J_j. Also it must be the case that

$$w_q(B'_j) \geq w_q(A_i) \text{ for every } i \in J_j, \tag{5.6}$$

for if $w_q(B'_j) < w_q(A_i)$ for some i and j we could exchange element $A_i+\mathcal{V}$ of basis $\mathcal{A}$ with $B'_j+\mathcal{V}$, contradicting the fact that $\mathcal{A}$ is a minimum q-weight basis. Using Equation

(5.4), we get

$$\begin{aligned}\pi(C)+\mathcal{V} &= b(B'_{-1}+\mathcal{V})+b(B'_0+\mathcal{V})+\sum_{j=0}^{n}(B'_j+\mathcal{V})\\ &= b\sum_{i\in J_{-1}}(A_i+\mathcal{V}) \;+\; b\sum_{i\in J_0}(A_i+\mathcal{V}) \;+\; \sum_{j=0}^{n}\sum_{i\in J_j}(A_i+\mathcal{V})\end{aligned}$$

Comparing this with (5.3) and using (5.5) and (5.6), it follows that $|C| \geq w_q(A_i) = |\widetilde{A}_i|$ for each $i \in I$.

Case 2. Suppose $|C| < q$. Then C must be a cycle in $G \times (C_q - e)$ for some edge $e \in E(C_q)$. By Lemma 3.3 there is an edge $ab \in E(C_q - e)$ for which

$$C = A + D \tag{5.7}$$

where $A \in \mathcal{C}(G \times ab) \subseteq \mathcal{C}(G \times C_q)$ and $D \in \mathcal{D}(G \times (C_q - e)) \subseteq \mathcal{D}(G \times C_q)$, and $|C| \geq |A|$.

We claim that $\pi(A)$ is an anti-cycle: WLOG assume $b = a+1$. Now, the degree of any vertex (x, a) of A is even, so let its neighbors be (y_i, b) for $1 \leq i \leq 2k$ for some integer k. Then the set of arcs of form $\pi((x, a)(y_i, b)) = \overrightarrow{xy_i}$ are the outward-pointing arcs at the vertex x of $\pi(A)$, so the out-degree of x is even. Similarly, since the degree of vertex (x, b) of A is even, the same argument shows that the in-degree of x is even. Thus $\pi(A)$ is an anti-cycle. Observe also that $|A| = |\pi(A)|$ because any arc $\overrightarrow{xy}$ of $\pi(A)$ can be the image of only one edge $(x, a)(y, b)$ of A. Since $|C| \geq |A|$, we also have $|C| \geq |\pi(A)| = w_q(\pi(A))$.

Write $\pi(A)+\mathcal{V}$ as

$$\pi(A)+\mathcal{V} = \sum_{j\in J}(A_j+\mathcal{V}), \tag{5.8}$$

and observe that we must have $w_q(\pi(A)) \geq w_q(A_j)$ for each $j \in J$, for otherwise some element $A_j+\mathcal{V}$ of the basis $\mathcal{A}$ could be exchanged for $\pi(A)+\mathcal{V}$, violating the fact that $\mathcal{A}$ is a minimal q-weight basis. Now take π' of both sides of Equation (5.2) to get $\pi'(C) = \sum_{i\in I}(A_i+\mathcal{V})$. Since $\pi'(C) = \pi'(A + D) = \pi'(A) = \pi(A)+\mathcal{V}$, we have $\pi(A)+\mathcal{V} = \sum_{i\in I}(A_i+\mathcal{V})$. Comparing this with Equation (5.8), we have $I = J$. Since we have established $|C| \geq w_q(\pi(A)) \geq w_q(A_j) = |\widetilde{A}_j|$ for all $j \in J$, we now also have $|C| \geq |\widetilde{A}_i|$ for all $i \in I$. This completes the proof. □

This concludes our solution to the special case $G \times C_q$ of the general problem of constructing an MCB for $G \times H$ in terms of invariants of the factors. We believe the reader may now have a sense of the complexity of such a general construction. We maintain hope that someone will find a simple construction for the general case, though we expect that such a construction would involve elements of our approach.

As noted in the introduction, the MCB problem has been resolved for the Cartesian, strong and lexicographic products. To our knowledge, the modular product (see Appendix C of [6]) is the only associative product that remains unexplored.

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