Pop-Up Geometry: The Mathematics Behind Pop-Up Cards

Reviewed By Richard H. Hammack

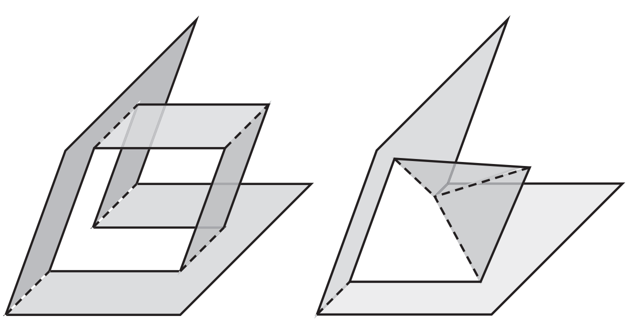
Are you are fascinated by the mathematics of paper-folding? Have you ever been awed by opening a pop-up card or book and seeing a three-dimensional configuration magically appear? If so, you will love the new book *Pop-Up Geometry* (Cambridge University Press, 2022) by Joseph O'Rourke.

This is O'Rourke's third book related to paper folding, following *How to Fold It* (2011) and *Geometric Folding Algorithms* with Erik Demaine (2007), and this one is accessible to anyone with an innate sense of geometry and a grasp of high school algebra. Its appeal is broad, though: Both students and research mathematicians have much to learn from it. Unlike most math textbooks, it can be read in a piecemeal fashion. A nugget here, a nugget there, and you will be on your way to understanding (and making!) remarkably complex constructions.

The book opens with the most basic pop-up mechanism, the *parallel fold*. This fold consists of two parallel cuts made in a paper rectangle that is folded on a vertical axis (the spine). The spine fold alternates valley–mountain–valley, producing a configuration like the one on the left of figure 1, which opens up into a box-like shape with rhombic sides.

Despite the parallel fold's simplicity, multiple (or iterated) applications and embellishments can result in very complex designs.

Figure 1. The parallel fold (left) and V-fold (right).



Next, the book explores the versatile V-fold, in which a single cut is made in a paper rectangle that is folded along a spine. Two additional fold lines join the two endpoints of the cut to a point on the spine, resulting in a configuration like the one shown on the right of figure 1. O'Rourke explains

in mathematical detail how the V-fold can be harnessed to yield various rotary motions and pop-up actions geared by the opening of the card.

Variants of the parallel fold and V-fold are the most common mechanisms in pop-up cards and books. So, by the end of the second chapter, you'll be ready to make your own constructions.

Chapters 3 and 4 focus on the *Knight's Visor* and a pop-up spinner. The Knight's Visor, which resembles the visor on a knight's helmet, is a lovely, easy-to-make structure that has a complex unfolding motion. O'Rourke connects this motion to the *nephroid* curve. The pop-up spinner provides another fascinating example, leading to an informative discussion of linkages and protein folding. Readers primarily interested in the pop-ups can skip the details and digressions if desired, but I found the connection to classical mathematics and other applications enlightening.

The next chapter investigates rigid origami and pop-up polytopes. Here the book becomes more open-ended, and O'Rourke mentions some tantalizing unsolved problems.

O'Rourke finishes the book with algorithmic issues. He describes two algorithms that produce pop-up designs for some restricted classes of structures and ends with an informal discussion of the fact that pop-up design is, in general, an NP-hard problem. Essentially, this means that there is probably no possible efficient algorithm that can generate a design for any arbitrary pop-up.

Using an engaging and conversational tone, O'Rourke strikes a nice balance between rigor and informality. It's easy to follow the details in the elementary material, yet also get the big picture for the more advanced topics. The book includes ample exercises, ranging from mathematical computations to card constructions, along with solutions. Whether you are a casual reader or a serious student of pop-ups, you will come away with new insights.

This book will open a world of possibilities. Get it and get to work. Happy folding! ●

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