Vertex-transitive direct products of graphs

Richard H. Hammack*
Department of Mathematics
Virginia Commonwealth University
Richmond, Virginia U.S.A.
rhammack@vcu.edu

Wilfried Imrich
Dept. of Mathematics and Information Technology
Montanuniversität Leoben
Leoben, Austria
wilfried.imrich@unileoben.ac.at

Submitted: May 3, 2017; Accepted: Apr 4, 2018; Published: Apr 27, 2018

Abstract

It is known that for graphs A and B with odd cycles, the direct product $A \times B$ is vertex-transitive if and only if both A and B are vertex-transitive. But this is not necessarily true if one of A or B is bipartite, and until now there has been no characterization of such vertex-transitive direct products. We prove that if A and B are both bipartite, or both non-bipartite, then $A \times B$ is vertex-transitive if and only if both A and B are vertex-transitive. Also, if A has an odd cycle and B is bipartite, then $A \times B$ is vertex-transitive if and only if both $A \times K_2$ and B are vertex-transitive.

Mathematics Subject Classifications: 05C76, 05C75

1 Introduction

Our graphs are finite, without multiple edges, but may have loops. The set of isomorphism classes of graphs that may have loops is denoted Γ_0, while Γ denotes those without loops. Thus $\Gamma \subset \Gamma_0$. We denote the complete graph on n vertices as K_n, whereas K_n^* is K_n with a loop added to each vertex. The automorphism group of a graph G is denoted as $\mathrm{Aut}(G)$. We say G is **vertex-transitive** if for any two vertices $x, y \in V(G)$, there is a $\varphi \in \mathrm{Aut}(G)$ for which $\varphi(x) = y$.

Recall that the **direct product** of two graphs A, B in Γ_0 or Γ is the graph $A \times B$ with vertices $V(A) \times V(B)$ and edges

$$E(A \times B) = \big\{(a,b)(a',b') \mid aa' \in E(A) \text{ and } bb' \in E(B)\big\}.$$

Figure 1 shows an example.

*Supported by Simons Foundation Collaboration Grant for Mathematicians 523748.

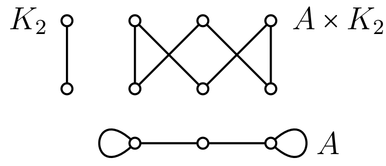

Figure 1: The direct product of graphs.

This product is commutative and associative in the sense that the maps $(x, y) \mapsto (y, x)$ and $(x, (y, z)) \mapsto ((x, y), z)$ are isomorphisms $A \times B \to B \times A$ and $A \times (B \times C) \to (A \times B) \times C$. If $+$ represents disjoint union, then the distributive law $A \times (B + C) = A \times B + A \times C$ holds, which is equality of graphs, rather than just isomorphism. Recall also Weichsel's theorem [4, Theorem 5.9].

Theorem 1. *A direct product $A \times B$ of connected graphs is connected if and only if at least one factor has an odd cycle; if both factors are bipartite, then the product has exactly two components. In general, if both A and B have odd cycles, then so does $A \times B$. Moreover, if B is bipartite, with bipartition $X \cup Y$, then $A \times B$ is bipartite, with bipartition $V(A) \times X \cup V(A) \times Y$.*

It is easy to verify that if B is bipartite, then $K_2 \times B = B + B$.

Note that $G \times K_1^* \cong G$ for any G, so K_1^* is the unit for the direct product. A graph G is called **prime** over the direct product if it has more than one vertex, and whenever $G \cong A \times B$, one of A or B is isomorphic to G, and the other is K_1^*. A result of McKenzie [6] implies that any finite, connected non-bipartite graph factors uniquely into prime graphs over the direct product, up to order and isomorphism of the factors. See also Chapter 8 of [4].

A consequence of unique prime factorization of connected non-bipartite graphs over the direct product, Theorem 8.19 of [4] states that any [non-bipartite] direct product is vertex-transitive if an only if each factor is vertex-transitive. Unfortunately, the condition of non-bipartiteness was inadvertently omitted in the statement of Theorem 8.19 [4]. Indeed, the theorem is false for non-bipartite graphs, as is seen in Figure 1, where $A \times K_2$ is the (vertex-transitive) 6-cycle, but A is not vertex-transitive.

Until now, no characterization of bipartite vertex-transitive direct products had been known. In Section 5 we give the following complete characterization.

Theorem 13. *If A and B are both bipartite or both non-bipartite, then $A \times B$ is vertex-transitive if and only if both A and B are vertex-transitive. If A has an odd cycle and B is bipartite, then $A \times B$ is vertex-transitive if and only if both $A \times K_2$ and B are vertex-transitive.*

One direction of this theorem is elementary, and follows from our next proposition.

Proposition 2. *If both A and B are vertex-transitive graphs, then $A \times B$ is vertex-transitive. If both $A \times K_2$ and B are vertex-transitive, and B is bipartite, then $A \times B$ is vertex-transitive.*

Proof. For the first statement, suppose both A and B are vertex-transitive. Given two vertices (a,b) and (a',b') of $A \times B$, select automorphisms α of A and β of B for which $\alpha(a) = a'$ and $\beta(b) = b'$. By the definition of the direct product, $(x,y) \mapsto (\alpha(x), \beta(y))$ is an automorphism of $A \times B$ sending (a,b) to (a',b'), so $A \times B$ is vertex-transitive.

For the second statement, suppose both $A \times K_2$ and B are vertex-transitive and B is bipartite. Then the above paragraph implies that $(A \times K_2) \times B$ is vertex-transitive. But

$$(A \times K_2) \times B \cong A \times (K_2 \times B) \cong A \times (B + B) = A \times B + A \times B,$$

which is to say that the graph consisting of two copies of $A \times B$ is vertex-transitive. Then certainly each copy of $A \times B$ is vertex-transitive. □

The converse of our main theorem is more subtle, and some machinery is needed to attack it. To this end, Section 2 reviews the Cartesian product of graphs, and their unique prime factorizations. This is followed by sections on R-thinness and Cartesian skeletons. Section 5 proves our main theorem. There we will need the Lovász cancellation laws:

Theorem 3 (Lovász [5])**.** *Suppose A, B and C are graphs, and C has at least one edge. Then $A \times C \cong B \times C$ implies $A \cong B$ provided that*

- *C has an odd cycle, or*
- *A and B are both bipartite.*

But before reviewing further preliminaries, some examples will put our results in context. Example 1 involves a factor with loops, Example 2 a disconnected factor without loops, and Example 3 involves a connected factor without loops.

Example 4. The graph A in Figure 1 is not vertex-transitive. However, the figure shows that $A \times K_2$ is vertex-transitive. If B is a bipartite vertex-transitive graph (such as the 4-cycle), then $A \times B$ is vertex-transitive by Proposition 2. So here we have a vertex-transitive product $A \times B$, where A is not vertex-transitive, but both $A \times K_2$ and B are.

Example 5. The graph $A = C_3 + C_6$ is not vertex-transitive because one component is a triangle and the other is a hexagon. But Figure 2 shows that $A \times K_2$ is three copies of a hexagon, which *is* vertex-transitive. If B is a bipartite vertex-transitive graph, then Proposition 2 says $A \times B$ is vertex-transitive. So $A \times B$ is vertex-transitive, where A is not vertex-transitive, but both $A \times K_2$ and B are.

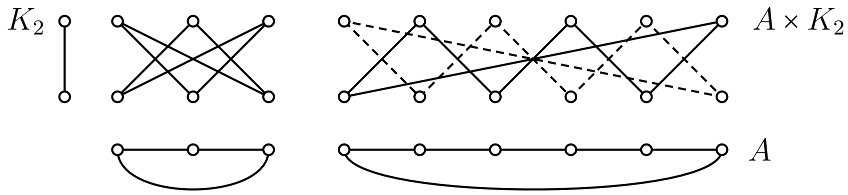

Figure 2: The disconnected graph A is not vertex-transitive, but its product with K_2 is.

Example 6. Figure 3 shows a graph A and the product $A \times K_2$. For brevity, vertices (x, ε) of $A \times K_2$ are written as $x\varepsilon$, and, for clarity, edges are encoded dashed, dotted, solid black and solid gray. The dashed outer hexagon H in A corresponds to a subgraph $H \times K_2 = H + H$ in the product, which is shown as two dashed copies of H. Similarly each solid (black or gray) triangle T in A corresponds to a solid (black or gray) hexagon $T \times K_2$ in the product.

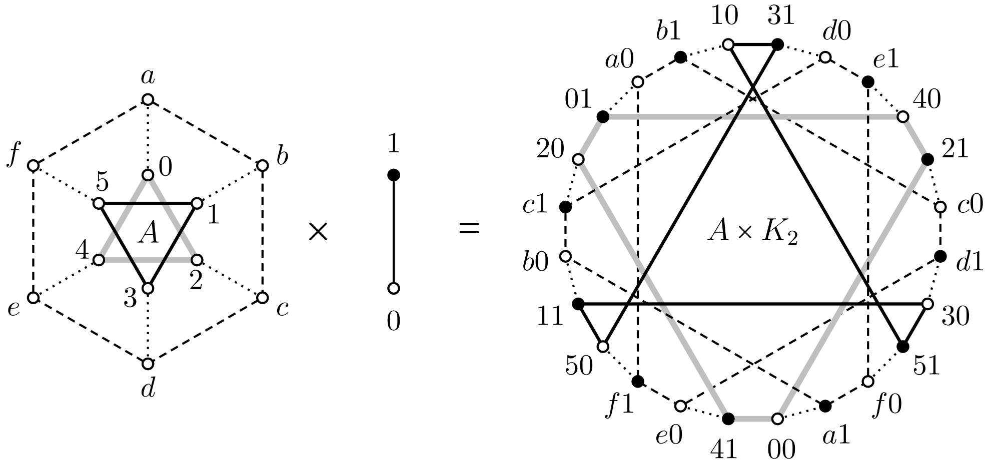

Figure 3: The graph A is not vertex-transitive, but its product with K_2 is.

The graph A is not vertex-transitive because some of its vertices are on triangles and others are not. But the product $A \times K_2$ *is* vertex-transitive, though seeing this may take a moment of reflection. Note that the permutation $\pi = (a\,b\,c\,d\,e\,f)(0\,1\,2\,3\,4\,5)$ that rotates A by 60° induces an automorphism $x\varepsilon \mapsto \pi(x)\varepsilon$ of $A \times K_2$ that sends the "twisted" solid black hexagon to the "untwisted" solid gray hexagon.

Also the following automorphism of order 2 exchanges the two dashed hexagons in the product with the two solid hexagons.

$$\begin{array}{cccccc} a0 & b1 & c0 & d1 & e0 & f1 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 01 & 40 & 21 & 00 & 41 & 20 \end{array} \qquad \begin{array}{cccccc} a1 & b0 & c1 & d0 & e1 & f0 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 30 & 11 & 50 & 31 & 10 & 51 \end{array}$$

The other symmetries are more transparent, arising from rotations and reflections of the product's drawing.

Now, if B is a bipartite vertex-transitive graph, then $A \times B$ is vertex-transitive by Proposition 2. So we have a vertex-transitive product $A \times B$, where A is not vertex-transitive, but both $A \times K_2$ and B are.

Now we cover the preliminary material needed to prove our main theorem, Theorem 13. We begin with the Cartesian product.

2 The Cartesian Product

The *Cartesian product* of two graphs $A, B \in \Gamma$ is the graph $A \,\square\, B \in \Gamma$ with vertices $V(A) \times V(B)$ and edges

$$E(A \,\square\, B) = \big\{(a,b)(a',b') \mid aa' \in E(A) \text{ and } b = b', \text{ or } a = a' \text{ and } bb' \in E(B)\big\}.$$

(See Figure 4.) The Cartesian product is commutative and associative in the sense that $A \,\square\, B \cong B \,\square\, A$ and $A \,\square\, (B \,\square\, C) \cong (A \,\square\, B) \,\square\, C$. Letting $B + C$ denote the disjoint union of graphs B and C, we also get the distributive law

$$A \,\square\, (B + C) = A \,\square\, B + A \,\square\, C, \tag{1}$$

which is true equality, rather than mere isomorphism.

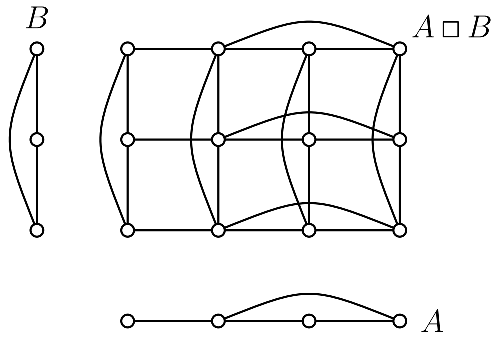

Figure 4: Cartesian product of graphs.

Clearly $K_1 \,\square\, A \cong A$ for any graph A, so K_1 is the unit for the Cartesian product. A nontrivial graph G is *prime over* $\square$ if for any factoring $G \cong A \,\square\, B$, one of A or B is K_1 and the other is G. Certainly every finite graph can be factored into prime factors in Γ. Sabidussi and Vizing [7, 8] proved that this prime factorization is unique for connected graphs. More precisely, we have the following.

Theorem 7 (Theorem 6.8 of [4]). *Let $G, H \in \Gamma$ be isomorphic connected graphs with $G = G_1 \,\square \cdots \square\, G_k$ and $H = H_1 \,\square \cdots \square\, H_\ell$, where the factors G_i and H_i are prime. Then $k = \ell$, and for any isomorphism $\varphi : G \to H$, there is a permutation π of $\{1, 2, \ldots, \ell\}$ and isomorphisms $\varphi_i : G_{\pi(i)} \to H_i$ for which*

$$\varphi(x_1, x_2, \ldots, x_\ell) = \big(\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_\ell(x_{\pi(\ell)})\big).$$

Notice that cancellation is a consequence of unique prime factorization: For connected graphs, $A \,\square\, C \cong B \,\square\, C$ implies $A \cong B$. (Cancellation holds also for disconnected graphs, but we shall not need this stronger result.)

3 R-Thin Graphs

The notion of so-called R-thinness is an important issue in factorings over the direct product. McKenzie [6] uses this idea (in a somewhat more general form), citing an earlier use by Chang [1]. In other contexts, R-thin graphs have been called *worthy* graphs [9]. See Chapter 8 of [4] for proofs of the assertions made in this section.

A graph G is *R-thin* if no two vertices have the same open neighborhood, that is, if $N_G(x) = N_G(y)$ implies $x = y$. Said differently, any vertex is uniquely determined by its open neighborhood.

More generally, we form a relation R on the vertices of an arbitrary graph. Two vertices x and y of G are in *relation* R, written xRy, precisely if their open neighborhoods are identical, that is, if $N_G(x) = N_G(y)$. It is easy to check that R is an equivalence relation on $V(G)$.

An R-equivalence class of a graph is called an **R-class**. Given two R-classes X and Y (not necessarily distinct), it is easy to check that either every vertex in X is adjacent to every vertex in Y, or no vertex in X is adjacent to any in Y. In particular, this means that an R-class X of G induces either a complete subgraph K_n^* or a totally disconnected subgraph $\overline{K_n^*}$.

As the relation R is defined entirely in terms of adjacencies, it is clear that given an isomorphism $\varphi : G \to H$ we have xRy in G if and only if $\varphi(x) R \varphi(y)$ in H. Thus φ maps R-classes of G bijectively to R-classes of H.

Take any vertex x of a vertex-transitive graph G, and say x belongs to the R-class X. Because an automorphism φ of G carries R-classes to R-classes, the R-class containing $\alpha(x)$ has $|X|$ vertices. Thus, by vertex-transitivity, all R-classes of G have size $|X|$.

Given a graph G, we define a quotient graph G/R (in Γ_0) whose vertex set is the set of R-classes of G, and for which two classes are adjacent if they are joined by an edge of G. (And a single class carries a loop provided that an edge of G has both endpoints in that class.) If G is R-thin, then $G/R \cong G$. An easy check confirms that G/R is R-thin for any $G \in \Gamma_0$.

Any automorphism $\varphi : G \to G$ induces an automorphism $G/R \to G/R$ defined as $X \mapsto \varphi(X)$. Conversely, if all R-classes have the same size, then we can lift any automorphism $\varphi : G/R \to G/R$ to an automorphism of G by simply declaring that each R-class X maps to $\alpha(X)$ by an arbitrary bijection. Moreover, if x and y are two vertices in the same R-class, then transposition of x with y is an automorphism of G. This implies a lemma.

Lemma 8. *If a graph G is vertex-transitive, then G/R is vertex-transitive. If G/R is vertex-transitive, and all R-classes of G have the same size, then G is vertex-transitive.*

Because $N_{A\times B}(a, b) = N_A(a) \times N_B(b)$, it follows that the R-classes of $A\times B$ are precisely the sets $X \times Y$, where X is an R-class of A, and Y is an R-class of B. In fact, the map $(A \times B)/R \to A/R \times B/R$ given by $X \times Y \mapsto (X, Y)$ is an isomorphism, as is proved in Section 8.2 of [4]. Thus $A \times B$ is R-thin if and only if both A and B are.

Note also that G is bipartite if and only if G/R is bipartite.

We will use these ideas frequently, sometimes without comment.

4 The Cartesian Skeleton

We now recall the definition of the Cartesian skeleton $S(G)$ of an arbitrary graph G in Γ_0. The Cartesian skeleton $S(G)$ is a graph on the vertex set of G that has the property $S(A \times B) = S(A) \,\square\, S(B)$ in the class of R-thin graphs, thereby linking the direct and Cartesian products.

We construct $S(G)$ as a certain subgraph of the Boolean square of G. The **Boolean square** of G is the graph G^s with $V(G^s) = V(G)$ and $E(G^s) = \{xy \mid N_G(x) \cap N_G(y) \neq \varnothing\}$. Thus, xy is an edge of G^s whenever G has an x, y-walk of length two. The left side of Figure 5 shows graphs A, B and $A \times B$ (bold) together with their Boolean squares A^s, B^s and $(A \times B)^s$ (dotted).

If G has an x, y-walk W of even length, then G^s has an x, y-walk of length $|W|/2$ on alternate vertices of W. Thus G^s is connected if G is connected and has an odd cycle. (An odd cycle guarantees an even walk between any two vertices.) On the other hand, if G is connected and bipartite, then G^s has exactly two components, whose vertex sets are the two partite sets of G.

We now show how to form $S(G)$ as a certain spanning subgraph of G^s. Consider an arbitrary factorization $G \cong A \times B$, by which we identify each vertex of G with an ordered pair (a, b). We say that an edge $(a, b)(a', b')$ of G^s is **Cartesian** relative to the factorization $A \times B$ if either $a = a'$ and $b \neq b'$, or $a \neq a'$ and $b = b'$. For example, in Figure 5 edges xz and zy of G^s are Cartesian (relative to the factorization $A \times B$), but edges xy and yy of G^s are not Cartesian. We make $S(G)$ from G^s by removing the edges of G^s that are not Cartesian, but we do this in a way that does not reference the factoring $A \times B$ of G. We next identify two intrinsic criteria for a non-loop edge of G^s that tell us if it may fail to be Cartesian relative to some factoring of G.

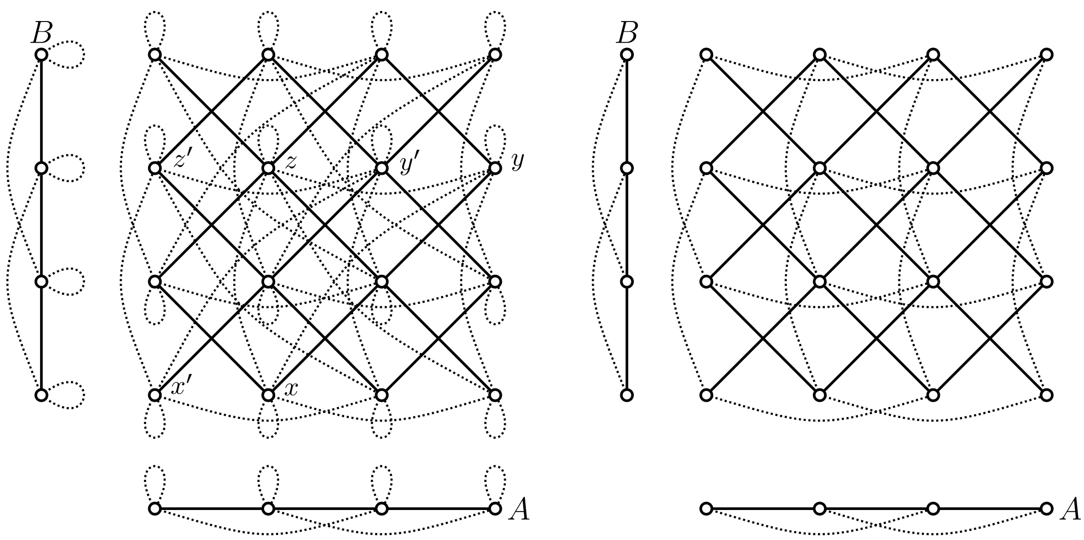

Figure 5: Left: graphs $A, B, A{\times}B$ and their Boolean squares A^s, B^s and $(A{\times}B)^s$ (dotted). Right: graphs $A, B, A{\times}B$ and their Cartesian skeletons $S(A), S(B)$ and $S(A{\times}B)$ (dotted).

The criteria are as follows. (Note that the symbol $\subset$ means *proper* inclusion, and the neighborhoods are neighborhoods of G, not G^s.)

(i) In Figure 5, the edge xy of G^s is not Cartesian, and there is a $z \in V(G)$ with $N_G(x) \cap N_G(y) \subset N_G(x) \cap N_G(z)$ and $N_G(x) \cap N_G(y) \subset N_G(y) \cap N_G(z)$.

(ii) In Figure 5, the edge $x'y'$ of G^s is not Cartesian, and there is a $z' \in V(G)$ with $N_G(x') \subset N_G(z') \subset N_G(y')$.

Our aim is to remove from G^s all edges that meet one of these criteria. We package the above criteria into the following definition. An edge xy of G^s is **dispensable** if $x = y$ or there exists $z \in V(G)$ for which both of the following statements hold.

(1) $N_G(x) \cap N_G(y) \subset N_G(x) \cap N_G(z)$ or $N_G(x) \subset N_G(z) \subset N_G(y)$,

(2) $N_G(y) \cap N_G(x) \subset N_G(y) \cap N_G(z)$ or $N_G(y) \subset N_G(z) \subset N_G(x)$.

Observe that the above statements (1) and (2) are symmetric in x and y. It is easy to check that (i) or (ii) holding for a triple x, y, z is equivalent to *both* of (1) and (2) holding. Now we come to this section's main definition. The **Cartesian skeleton** of a graph G is the spanning subgraph $S(G)$ of G^s obtained by removing all dispensable edges from G^s.

The right side of Figure 5 is the same as its left side, except all dispensable edges of A^s, B^s and $(A \times B)^s$ are deleted. Thus the remaining dotted edges are $S(A), S(B)$ and $S(A \times B)$. Note that although $S(G)$ was defined without regard to the factorization $G = A \times B$, we nonetheless have $S(A \times B) = S(A) \,\square\, S(B)$. The following proposition from [3] asserts that this always holds for R-thin graphs.

Proposition 9. *If A, B are R-thin graphs, then $S(A \times B) = S(A) \,\square\, S(B)$, provided that neither A nor B has any isolated vertices. This is equality, not mere isomorphism; the graphs $S(A \times B)$ and $S(A) \,\square\, S(B)$ have identical vertex and edge sets.*

As $S(G)$ is defined entirely in terms of the adjacency structure of G, we have the following immediate consequence.

Proposition 10. *Any isomorphism $\varphi : G \to H$, as a map $V(G) \to V(H)$, is also an isomorphism $\varphi : S(G) \to S(H)$.*

We will also need a result concerning connectivity of Cartesian skeletons. The following result (which does not require R-thinness) is from [3]. (For another proof, see Chapter 8 of [4].)

Proposition 11. *Suppose G is connected.*

(i) *If G has an odd cycle, then $S(G)$ is connected.*

(ii) *If G is nontrivial bipartite, then $S(G)$ has two connected components. Their respective vertex sets are the two partite sets of G.*

5 Main Result

The next proposition is the technical heart of this paper, and uses the material of the previous three sections. It is followed by (and implies part of) Theorem 13, our main characterization of vertex-transitive direct products.

Proposition 12. *Let A and B be R-thin, connected graphs, either both non-bipartite, or both bipartite. If $A \times B$ is vertex-transitive, then both A and B are vertex-transitive.*

Proof. The proof has two parts. Part 1 proves the result assuming that A and B are both non-bipartite. Part 2 proves it if they are both bipartite.

Part 1. Assume A and B are non-bipartite and $A \times B$ is vertex-transitive. We will prove that A is vertex-transitive. (The same reasoning works for B.) In what follows, $a, a' \in V(A)$ are two arbitrary vertices. We will construct $\theta \in \mathrm{Aut}(A)$ with $\theta(a) = a'$.

Fix $b \in V(B)$ and select $\varphi \in \mathrm{Aut}(A \times B)$ with $\varphi(a,b) = (a',b)$. By Proposition 10, φ is also an automorphism $\varphi : S(A \times B) \to S(A \times B)$, and by Proposition 9, it is an automorphism $\varphi : S(A) \,\square\, S(B) \to S(A) \,\square\, S(B)$. By Proposition 11, $S(A)$ and $S(B)$ are connected. Form prime factorizations $S(A) = G_1 \,\square \cdots \square\, G_k$, and $S(B) = G_{k+1} \,\square \cdots \square\, G_\ell$, so

$$S(A) \,\square\, S(B) \quad = \quad \overbrace{G_1 \,\square \cdots \square\, G_k}^{S(A)} \,\square\, \overbrace{G_{k+1} \,\square \cdots \square\, G_\ell}^{S(B)},$$

We've now coordinatized $S(A)$ and $S(B)$ (hence also A and B) so that

$$a = (a_1, \dots, a_k), \quad a' = (a'_1, \dots, a'_k), \quad b = (b_{k+1}, \dots, b_\ell)$$

for tuples of $a_i, a'_i \in V(G_i)$ $(1 \le i \le k)$, and $b_i \in V(G_i)$ $(k+1 \le i \le \ell)$. Furthermore, φ is

$$\varphi : (G_1 \,\square \cdots \square\, G_k) \,\square\, (G_{k+1} \,\square \cdots \square\, G_\ell) \quad \longrightarrow \quad (G_1 \,\square \cdots \square\, G_k) \,\square\, (G_{k+1} \,\square \cdots \square\, G_\ell.) \qquad (2)$$

Theorem 7 applied to (2) says there is a permutation π of $\{1, 2, \dots, \ell\}$ and isomorphisms $\varphi_i : G_{\pi(i)} \to G_i$ for which

$$\varphi\big((x_1, \dots, x_k), (x_{k+1}, \dots, x_\ell)\big) = \Big(\big(\varphi_1(x_{\pi(1)}), \dots, \varphi_k(x_{\pi(k)})\big), \big(\varphi_{k+1}(x_{\pi(k+1)}), \dots, \varphi_\ell(x_{\pi(\ell)})\big)\Big). \ (3)$$

If we are lucky, then π permutes the indices $\{1, \dots, k\}$ among themselves, and $\{k+1, \dots, \ell\}$ among *themselves*. We can then define $\theta : A \to A$, as

$$\theta(x_1, \dots, x_k) \quad = \quad \big(\varphi_1(x_{\pi(1)}), \, \varphi_2(x_{\pi(2)}), \, \dots, \varphi_k(x_{\pi(k)})\big).$$

It is easy to check (and is proved below) that θ is an automorphism of A sending $a = (a_1, \dots, a_k)$ to $a' = (a'_1, \dots, a'_k)$, as desired.

But in general, π will not respect the indices like this, and constructing θ involves more care. In general, if we decompose π into disjoint cycles, some of them may interchange some indices in $\{1, \dots, k\}$ with those in $\{k+1, \dots, \ell\}$. Figure 6 illustrates a typical such cycle $\big(i \; \pi(i) \; \pi^2(i) \; \dots \; \pi^6(i)\big)$ of length 7.

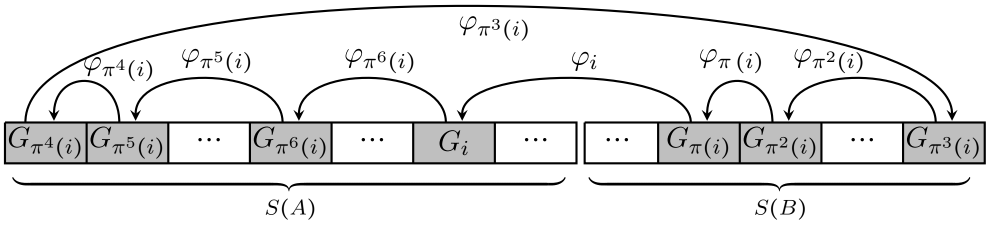

Figure 6: Effect of φ on coordinates corresponding to a typical cycle of π

To control such mixing of coordinates, we define a **stacking operation** on $V(A\times B)$. This will allow us to modify φ so its $S(A)$ coordinate functions do not depend on the factors of $S(B)$. The idea, introduced in [2], takes an input vertex $(x,y_0)\in V(A\times B)$, applies φ to get $\varphi(x,y_0)=(x_1,y_1)$, then replaces the x_1 with x.

Stacking Operation

0. Begin with input vertex $(x,\ y_0)\ \in\ V\big(S(A)\,\square\,S(B)\big)=V(A\times B)$
1. Apply φ: $(x_1,y_1)\ \in\ V\big(S(A)\,\square\,S(B)\big)=V(A\times B)$
2. Replace x_1 with x: $(x,\ y_1)\ \in\ V\big(S(A)\,\square\,S(B)\big)=V(A\times B)$

The stacking operation sends a vertex (x,y_0) to an output (x,y_1), to which we can again apply the stacking operation to get (x,y_2), etc. This process yields a sequence

$$(x,y_0),(x,y_1),(x,y_2),(x,y_3),\ldots. \tag{4}$$

For example, let's trace this sequence with $x=(r,s,\ldots,t,\ldots,u,\ldots)$, and where we only consider those coordinates which correspond to the cycle of π indicated in Figure 6. The first five terms are as follows, where for typographical efficiency we use the abbreviations $\bar{r}=\varphi_{\pi^3(i)}(r)$ as well as $\bar{\bar{r}}=\varphi_{\pi^2(i)}\circ\varphi_{\pi^3(i)}(r)$ and $\bar{\bar{\bar{r}}}=\varphi_{\pi(i)}\circ\varphi_{\pi^2(i)}\circ\varphi_{\pi^3(i)}(r)$.

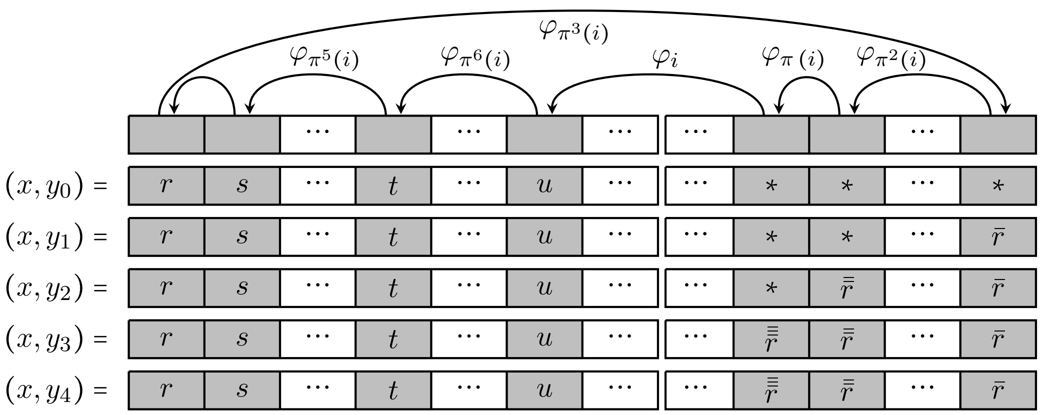

Figure 7: The effect of the stacking operation on a typical cycle of π.

By the third iteration, the coordinates of y_0 in this cycle have been "flushed out" and replaced (or "stacked") with images of r. Also, further iterations affect no further changes on this cycle. For $M \geq \ell - k$, the terms (x, y_M) of Sequence 4 agree on all cycles that permute at least one vertex of $\{1, \ldots, k\}$. (A cycle permuting only indices in $\{k+1, \ldots, \ell\}$ has not been flushed out.)

Consider now what happens when we apply φ to the stacked vertex (x, y_M). This is shown in Figure 8, with (x, y_0) as in the bottom of Figure 7. Notice that $\varphi(x, y_M) = \big(\theta(x), \eta(y_M)\big)$, where $\theta : V(A) \to V(A)$ is a bijection and $\eta(y_M)$ is some vertex of $S(B)$.

Remark 1: Note that η need not be the identity. If π has a non-trivial cycle that permutes only indices in $\{k+1, \ldots, \ell\}$, then the corresponding coordinates of y_M do not get stacked, so φ may alter these coordinates.

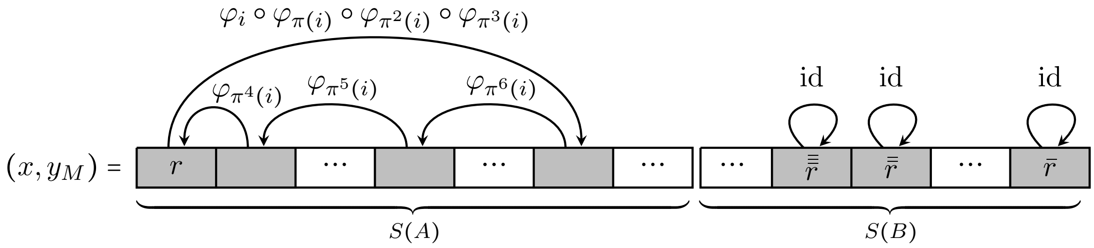

Figure 8: The effect of φ on a stacked vertex (x, y_M).

Remark 2: Our cycle $\big(\pi^4(i), \pi^5(i), \pi^6(i), i, \pi(i), \pi^2(i), \pi^3(i)\big)$ of figures 6, 7 and 8 has a string of entries between 1 and k, followed by a string between $k+1$ and ℓ. In general a cycle of π may alternate between these two types of strings numerous times. Notice that in such a case coordinates of $S(B)$ still get stacked with images coordinates of $S(A)$.

Since $\varphi(a, b) = (a', b)$, the stacking operation does not alter (a, b). After M iterations we still have $(a, b_M) = (a, b)$, so $(a', b) = \varphi(a, b) = \varphi(a, b_M) = (\theta(a), \eta(b_M))$. This implies

$$\theta(a) = a'. \tag{5}$$

To complete Part 1, we just need to show $\theta \in \mathrm{Aut}(A)$. Take $xy \in E(A)$. We claim $\theta(x)\theta(y) \in E(A)$. Fix $bb' \in E(B)$, so that $(x, b)(y, b') \in E(A \times B)$. Apply the stacking operation on the two endpoints, in parallel.

0.	Begin with input edge:	$(x, b)(y, b')$	$\in$	$E(A \times B)$
1.	Apply φ to endpoints:	$(x_1, b_1)(y_1, b_1')$	$\in$	$E(A \times B)$
2.	Replace x_1 with x, and y_1 with y:	$(x, b_1)(y, b_1')$	$\in$	$E(A \times B)$

After $M \geq \ell - k$ iterations, we have $(x, b_M)(y, b_M') \in E(A \times B)$. Applying φ to both endpoints gives $\big(\theta(x), \eta(b_M)\big)\big(\theta(y), \eta(b_M')\big) \in E(A \times B)$. Thus $\theta(x)\theta(y) \in E(A)$, meaning $\theta : A \to A$ is a bijective homomorphism, hence also an automorphism because A is finite.

In summary, $\theta \in \mathrm{Aut}(A)$, and $\theta(a) = a'$ by Equation (5). This means A is vertex-transitive. Because the direct product is commutative, it follows that the other factor B is also vertex transitive.

Part 2. Assume A and B are bipartite and $A \times B$ is vertex-transitive. We will show A is vertex-transitive. (And thus so is B, by commutativity of $\times$.) To begin, we analyze skeletons. Proposition 11 implies $S(A) = A_0 + A_1$ is the disjoint union of two graphs whose respective vertex sets are the partite sets of A. Similarly $S(B) = B_0 + B_1$. Thus

$$\begin{aligned} S(A \times B) = S(A) \,\square\, S(B) &= (A_0 + A_1) \,\square\, (B_0 + B_1) \\ &= A_0 \,\square\, B_0 + A_0 \,\square\, B_1 + A_1 \,\square\, B_0 + A_1 \,\square\, B_1. \end{aligned}$$

This is illustrated in Figure 9. By Weichsel's theorem, $A \times B$ is bipartite and has two components. One component has as partite sets the vertices of $A_0 \,\square\, B_0$ and $A_1 \,\square\, B_1$, and the other component's partite sets are the vertices of $A_0 \,\square\, B_1$ and $A_1 \,\square\, B_0$.

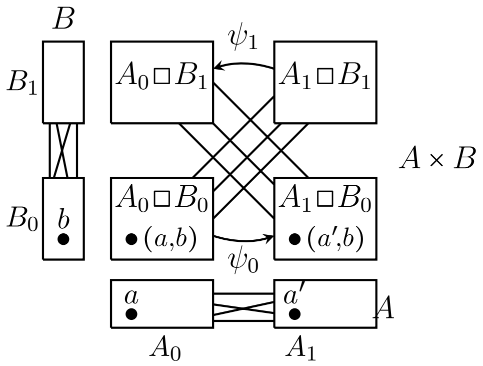

Figure 9: The effect of φ on a direct product of two connected bipartite graphs A and B.

As $A \times B$ is vertex-transitive, it has automorphisms mapping any of its partite sets to any other, so by Proposition 10, the four skeleton components $A_0 \,\square\, B_0$, $A_0 \,\square\, B_1$, $A_1 \,\square\, B_0$ and $A_1 \,\square\, B_1$ are all isomorphic to one other. By cancellation, $A_0 \cong A_1$ and $B_0 \cong B_1$.

Let $a, a' \in V(A)$. We will produce an automorphism θ of A with $\theta(a) = a'$. Notice that it suffices to prove this for a and a' in different partite sets of A. For if this is established and a, a' are in the same partite set, then we can take an a'' in the opposite partite set. The composition of two automorphisms mapping $a \mapsto a''$ and $a'' \mapsto a'$ is an automorphism of A mapping a to a'. Thus assume $a \in V(A_0)$ and $a' \in V(A_1)$. Fix some $b \in V(B_0)$, and let φ be an automorphism of $A \times B$ with $\varphi(a, b) = (a', b)$. (See Figure 9.) From this we will now construct $\theta \in \mathrm{Aut}(A)$ with $\theta(a) = a'$.

The map φ restricts to isomorphisms $\psi_1 : A_1 \square B_1 \to A_0 \square B_1$ and $\psi_0 : A_0 \square B_0 \to A_1 \square B_0$, which, together, send one component of $A \times B$ to the other. (See Figure 9.)

Prime factor A_0 as $A_0 = G_1 \,\square \cdots \square\, G_k$, and B_0 as $B_0 = G_{k+1} \,\square \cdots \square\, G_\ell$. As $A_1 \cong A_0$ and $B_1 \cong B_0$, we can label the vertices of A_1 and B_1 so that they also have prime factorizations $A_1 = G_1 \,\square \cdots \square\, G_k$ and $B_1 = G_{k+1} \,\square \cdots \square\, G_\ell$. Then

$$A_0 \square B_1 = \overbrace{G_1 \,\square \cdots \square\, G_k}^{A_0} \,\square\, \overbrace{G_{k+1} \,\square \cdots \square\, G_\ell}^{B_1}, \quad A_1 \square B_1 = \overbrace{G_1 \,\square \cdots \square\, G_k}^{A_1} \,\square\, \overbrace{G_{k+1} \,\square \cdots \square\, G_\ell}^{B_1},$$

$$A_0 \square B_0 = \overbrace{G_1 \,\square \cdots \square\, G_k}^{A_0} \,\square\, \overbrace{G_{k+1} \,\square \cdots \square\, G_\ell}^{B_0}, \quad A_1 \square B_0 = \overbrace{G_1 \,\square \cdots \square\, G_k}^{A_1} \,\square\, \overbrace{G_{k+1} \,\square \cdots \square\, G_\ell}^{B_0},$$

and our restricted isomorphisms ψ_1 and ψ_0 are

$$\psi_1 : (G_1 \square \cdots \square G_k) \,\square\, (G_{k+1} \square \cdots \square G_\ell) \longrightarrow (G_1 \square \cdots \square G_k) \,\square\, (G_{k+1} \square \cdots \square G_\ell), \tag{6}$$
$$\psi_0 : (G_1 \square \cdots \square G_k) \,\square\, (G_{k+1} \square \cdots \square G_\ell) \longrightarrow (G_1 \square \cdots \square G_k) \,\square\, (G_{k+1} \square \cdots \square G_\ell). \tag{7}$$

We have now coordinatized A_0, A_1, B_0 and B_1 (hence also $A \times B$) so that

$$a = (a_1, \dots, a_k), \quad a' = (a'_1, \dots, a'_k), \quad b = (b_{k+1}, \dots, b_\ell)$$

for tuples of $a_i, a'_i \in V(G_i)$ $(1 \le i \le k)$, and $b_i \in V(G_i)$ $(k+1 \le i \le \ell)$.

Theorem 7 applied to (6) and (7) yields permutations π and σ of $\{1, \dots, \ell\}$ and isomorphisms $\varphi_i : G_{\pi(i)} \to G_i$ and $\varphi'_i : G_{\sigma(i)} \to G_i$ so that

$$\psi_1\big((x_1, \dots x_k), (x_{k+1}, \dots, x_\ell)\big) = \big((\varphi_1(x_{\pi(1)}), \dots, \varphi_k(x_{\pi(k)})), (\varphi_{k+1}(x_{\pi(k+1)}), \dots, \varphi_\ell(x_{\pi(\ell)}))\big), \tag{8}$$
$$\psi_0\big((x_1, \dots x_k), (x_{k+1}, \dots, x_\ell)\big) = \big((\varphi'_1(x_{\sigma(1)}), \dots, \varphi'_k(x_{\sigma(k)})), (\varphi'_{k+1}(x_{\sigma(k+1)}), \dots, \varphi'_\ell(x_{\sigma(\ell)}))\big). \tag{9}$$

Consider the effect of ψ_1 on a typical cycle of π, say a cycle $(i, \pi(i), \pi^2(i), \dots, \pi^6(i))$ of length 7, similar to that of Figure 6 (except that this time the map is $A_1 \square B_1 \to A_0 \square B_1$ rather than $S(A) \square S(B) \to S(A) \square S(B)$). This is represented schematically in Figure 10.

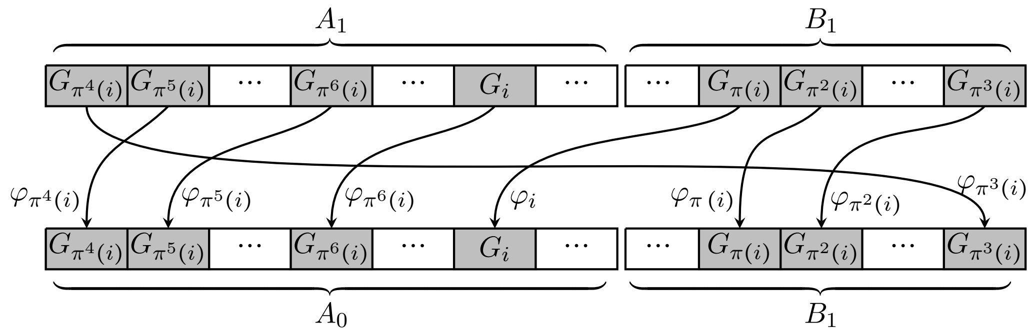

Figure 10: The effect of ψ_1 on coordinates corresponding to a typical cycle of π.

Notice that when we apply the stacking operation to a vertex $(x, y_0) \in V(A_1 \square B_1)$, we arrive at a vertex (x, y_1) that is still in $V(A_1 \square B_1)$. Furthermore, iterations of the stacking operation on (x, y_0) are exactly as indicated in Figure 7, and we get a sequence

$$(x, y_0), \ (x, y_1), \ (x, y_2), \ \dots, \ (x, y_M) \ \text{ in } A_1 \square B_1.$$

As in the first part of the proof, if $M \ge \ell - k$, then, in general, any jth coordinate of y_M for which $\pi^s(j) \le k$ (for some s) has been flushed out and replaced with the image of a vertex of some G_i with $1 \le i \le k$. It follows that $\psi_1(x, y_M) = \big(\theta_1(x), \eta_1(y_M)\big)$, where $\theta_1 : V(A_1) \to V(A_0)$ is a bijection. (This is illustrated in Figure 11.)

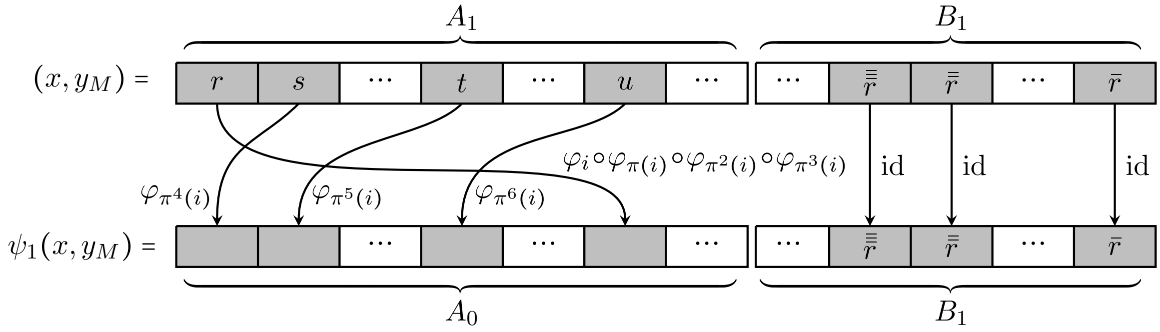

Figure 11: The effect of ψ_1 on a stacked vertex.

Likewise the stacking operation applied to a vertex $(w, z_0) \in V(A_0 \square B_0)$ yields a vertex $(w, z_1) \in V(A_0 \square B_0)$. Applying the stacking operation iteratively to (w, z_0), gives a sequence

$$(w, z_0),\ (w, z_1),\ (w, z_2),\ \ldots,\ (w, z_M) \quad \text{in } A_0 \square B_0.$$

As before, $\psi_0(w, z_M) = \big(\theta_0(w), \eta_0(z_M)\big)$, for some bijection $\theta_0 : V(A_0) \to V(A_1)$.

Let $\theta : A \to A$ be the map that restricts to θ_0 on $V(A_0)$ and θ_1 on $V(A_1)$. Thus θ reverses the bipartition of A. Because $\varphi(a, b) = (a', b)$, the stacking operation does not alter (a, b). That is, applying it M times to (a, b) yields (a, b_M) with $b_M = b$. Therefore

$$(a', b) = \varphi(a, b) = \varphi(a, b_M) = \psi_0(a, b_M) = (\theta_0(a), \sigma_0(a, b_M)),$$

which means $\theta_0(a) = a'$, that is, $\theta(a) = a'$.

To complete the proof we need to show $\theta \in \mathrm{Aut}(A)$. For this, take an edge $wx \in E(A)$, with $w \in V(A_0)$ and $x \in V(A_1)$. We claim $\theta(w)\theta(x) \in E(A)$. Fix an edge $z_0 y_0$ of B with $z_0 \in V(B_0)$ and $y_0 \in V(B_1)$, so $(w, z_0)(x, y_0)$ is an edge of $A \times B$ with $(w, z_0) \in V(A_0 \square B_0)$ and $(x, y_0) \in V(A_1 \square B_1)$. Apply the stacking operation on the two endpoints, in parallel.

0.	Begin with input edge:	$(w, z_0)(x, y_0)$	$\in$	$E(A \times B)$
1.	Apply φ (ψ_0 on left, ψ_1 on right):	$(w_1, z_1)(x_1, y_1)$	$\in$	$E(A \times B)$
2.	Replace w_1 with w, and x_1 with x:	$(w, z_1)(x, y_1)$	$\in$	$E(A \times B)$

Iterating this M times produces an edge $(w, z_M)(x, y_M) \in E(A \times B)$. Applying φ to both endpoints, we get

$$\big(\theta_0(w), \sigma_0(z_M)\big)\big(\theta_1(x), \sigma_1(y_M)\big) \in E(A \times B).$$

From this, $\theta_0(w)\theta_1(x) \in E(A)$, meaning $\theta(w)\theta(x) \in E(A)$. Thus $\theta : A \to A$ is a bijective homomorphism, hence also an automorphism because A is finite. As $\theta(a) = a'$, the graph A is vertex transitive. The proof is complete. □

We now reach our main theorem.

Theorem 13. *If A and B are both non-bipartite or both bipartite, then $A \times B$ is vertex-transitive if and only if both A and B are vertex-transitive. If A has an odd cycle and B is bipartite, then $A \times B$ is vertex-transitive if and only if both $A \times K_2$ and B are vertex-transitive.*

Proof. If both A and B are vertex-transitive, then Proposition 2 implies $A \times B$ is vertex-transitive. By the same proposition, if both $A \times K_2$ and B are vertex-transitive and B is bipartite, then $A \times B$ is vertex-transitive (as $A \times K_2$ is bipartite). Thus it remains to prove the converses of the two statements.

First, suppose A and B are non-bipartite, and $A \times B$ is vertex-transitive. Write $A = A_1 + \cdots + A_m$ and $B = B_1 + \cdots + B_m$ as disjoint unions of their components, with A_1 and B_1 non-bipartite. Then $A \times B = \sum_{i,j} A_i \times B_j$. By Weichsel's theorem, $A_i \times B_1$ and $A_j \times B_1$ are connected for any i, j, so these are two components of $A \times B$. But as $A \times B$ is vertex-transitive, all its components are isomorphic, so $A_i \times B_1 \cong A_j \times B_1$ and $A_i \cong A_j$ by Theorem 3. Thus any two components of A are isomorphic (and non-bipartite). By the same argument, any two components of B are isomorphic (and non-bipartite).

In the previous paragraph we remarked that $A_1 \times B_1$ is vertex-transitive. By Lemma 8, $(A_1 \times B_1)/R$ is vertex-transitive, and it is R-thin by the remarks in Section 3. Because $(A_1 \times B_1)/R \cong A_1/R \times B_1/R$, and both A_1/R and B_1/R are non-bipartite, Proposition 12 says A_1/R and B_1/R are vertex-transitive.

Now, because all R-classes of $A \times B$ have the same size, and each R-class of $A \times B$ is a product of R-classes of A and B, respectively, we conclude that all R-classes of A (hence of A_1) have the same size, and all R-classes of B (hence of B_1) have the same size. Lemma 8 now implies that A_1 and B_1 are vertex-transitive. But the components of A are all isomorphic to A_1, so A is vertex-transitive. Likewise, so is B.

Next suppose A and B are both bipartite, and $A \times B$ is vertex-transitive. Again, $A \times B = \sum_{i,j} A_i \times B_j$, and by Weichsel's theorem, each summand $A_i \times B_i$ has exactly two components. Because $A \times B$ is vertex-transitive, its components are all isomorphic, and vertex-transitive themselves, and it follows that each $A_i \times B_i$ is vertex-transitive.

Thus $A_i \times B_1 \cong A_j \times B_1$ for any i, j, so $A_i \cong A_j$ by Theorem 3, that is, all components of A are isomorphic. Similarly, all components of B are isomorphic.

By Lemma 8, $(A_1 \times B_1)/R$ is vertex-transitive, and it is also R-thin. As $(A_1 \times B_1)/R \cong A_1/R \times B_1/R$, and A_1/R and B_1/R are both bipartite, Proposition 12 implies that both A_1/R and B_1/R are vertex-transitive.

From here, the proof proceeds exactly as in the non-bipartite case, and we conclude that both A_1 and B_1 are vertex-transitive, thus also A and B.

For the second statement's converse, let A be non-bipartite, B bipartite, and $A \times B$ vertex-transitive. By Proposition 2, $(A \times B) \times K_2 \cong (A \times K_2) \times B$ is vertex-transitive. As $A \times K_2$ and B are bipartite, the first statement of the theorem (proved above) implies that $A \times K_2$ and B are vertex-transitive. □

References

[1] C.C. Chang, Cardinal factorization of finite relational structures, *Fund. Math.* **60** (1967) 251–269.

[2] C. Crenshaw and R. Hammack, Edge-transitive bipartite direct products, preprint.

[3] R. Hammack and W. Imrich, On Cartesian skeletons of graphs, *Ars Math. Contemp.*, **2** (2009) 191–205.

[4] *Handbook of Product Graphs,* second edition, by R. Hammack, W. Imrich and S. Klavžar, Series: Discrete Mathematics and Its Applications, CRC Press, Boca Raton, FL, 2011.

[5] L. Lovász, *On the cancellation law among finite relational structures*, Period. Math. Hungar. **1**(2) (1971) 145–156.

[6] R. McKenzie, Cardinal multiplication of structures with a reflexive relation, *Fund. Math.* **70** (1971) 59–101.

[7] G. Sabidussi, Graph multiplication, *Math. Z.*, **72** (1960) 446–457.

[8] G.V. Vizing, The Cartesian product of graphs, *Vyčisl Sistemy*, **9** (1963) 30–43.

[9] S. Wilson, A worthy family of semisymmetric graphs, *Discrete Math.*, **271** (2003) 283–294.