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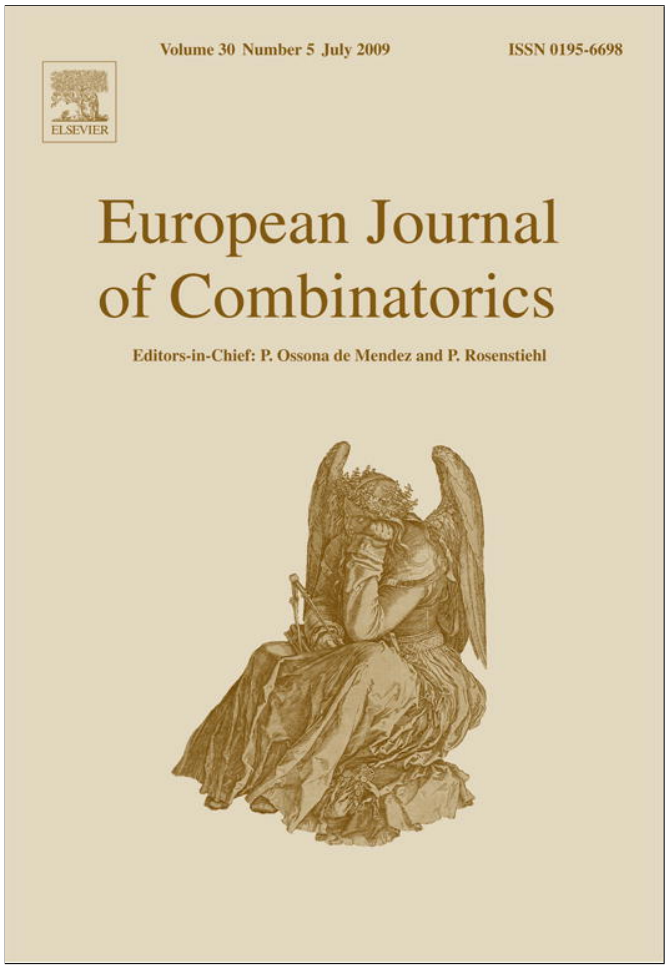

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Proof of a conjecture concerning the direct product of bipartite graphs

Richard H. Hammack

Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, VA 23284-2014, USA

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ABSTRACT

We prove that if the direct product of two connected bipartite graphs has isomorphic components, then one of the factors admits an automorphism that interchanges its partite sets. This proves a conjecture made by Jha, Klavžar and Zmazek in 1997 [P. Jha, S. Klavzar, B. Zmazek, Isomorphic components of Kronecker product of bipartite graphs, Discussiones Mathematicae Graph Theory 17 (1997) 302–308].

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1. Introduction

If G and H are graphs (possibly with loops) then their *direct product* is the graph $G \times H$ whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edges are all pairs $(g, h)(g', h')$ with $gg' \in E(G)$ and $hh' \in E(H)$.

It is a standard fact, first proved by Weichsel [8], that if G and H are connected and bipartite, then $G \times H$ has exactly two components. These components may or may not be isomorphic, depending on G and H. For example, Fig. 1 shows two products of bipartite graphs, where in each case the two components are distinguished by solid and dashed lines. In Fig. 1(a) the components are not isomorphic, and in Fig. 1(b) they are.

A number of authors have sought structural conditions on G and H that characterize the condition of $G \times H$ having isomorphic components. Jha, Klavžar and Zmazek [6] observed that this condition seems to be related to a certain kind of symmetry in at least one of the factors. They prove that if either G or H admits an automorphism that interchanges its partite sets, then $G \times H$ has isomorphic components. They conjecture that the converse is true. In [3] it is proved that the converse is true if G and H are square-free. This note presents a general proof of the converse.

To summarize our main result, we state the following definition and theorem. (The definition was introduced in [6].)

E-mail address: rhammack@vcu.edu.

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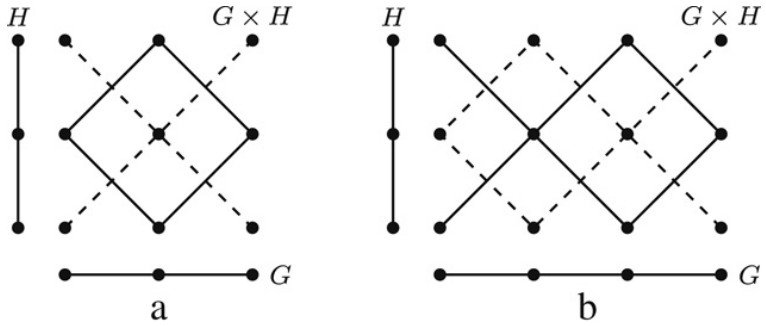

Fig. 1. Products of bipartite graphs.

Definition 1. A connected bipartite graph has *property* π if it admits an automorphism that interchanges its partite sets.

Theorem 1. *Suppose G and H are connected bipartite graphs. The two components of* $G \times H$ *are isomorphic if and only if at least one of* G *or* H *has property* π.

As an example, neither G nor H in Fig. 1(a) has property π, and indeed the components of $G \times H$ are not isomorphic. By contrast, in Fig. 1(b) G has property π (the nontrivial automorphism switches the bipartition) and the components of $G \times H$ are isomorphic.

One direction of Theorem 1 is proved as follows. Suppose G or H (say G) has property π. Then there is an automorphism φ of G that interchanges its partite sets. By [6] (Theorem 3.2), the map given by $(g, h) \mapsto (\varphi(g), h)$ restricts to an isomorphism of one component of $G \times H$ to the other, so if one of G or H has property π, then $G \times H$ has isomorphic components. The remainder of this note is devoted to proving the converse.

2. Preliminaries

We assume that the reader is familiar with the basic properties of direct products and graph homomorphisms. (For excellent surveys, see [5,4].) For convenience, we collect in this section some necessary definitions, ideas and standard results.

Observe that the direct product can be regarded as a product on digraphs. Given a digraph G, we let gg' denote an arc pointing from g to g'. Then $G \times H$ is the digraph with arcs $(g, h)(g', h')$ directed from (g, h) to (g', h') whenever there are arcs gg' in G and hh' in H. Since any graph can be identified with a symmetric digraph (where each edge is replaced by a double arc) the direct product of graphs is a special case of the direct product of digraphs. Though our main result is about graphs, we will use digraphs where necessary in the proofs.

Suppose G is a digraph whose vertices are ordered as $g_1, g_2, \ldots, g_m$. The *adjacency matrix* for G relative to this ordering is the $m \times m$ matrix A for which $a_{ij} = 1$ or 0 according to whether $g_i g_j$ is or is not an arc of G. Graphs are identified with symmetric digraphs, so the adjacency matrix of a graph is symmetric, that is it satisfies $A^{\mathrm{T}} = A$, where T indicates transpose.

The *tensor product* of two matrices A and B is the matrix $A \otimes B$ obtained by replacing each entry a_{ij} of A with the block $a_{ij}B$. The rows and columns of $A \otimes B$ thus divide into blocks corresponding to the rows and columns of A. The rows (columns) of $A \otimes B$ are indexed by ordered pairs so that (i, j) indexes the row (column) corresponding to the jth row (column) of B in the ith row-block (column-block). Therefore the entry of $A \otimes B$ in row (i, j) and column (k, ℓ) equals $a_{ik}b_{j\ell}$. Although $A \otimes B \neq B \otimes A$ in general, there are permutation matrices M and N for which $M(A \otimes B)N = B \otimes A$. (Matrix M corresponds to the permutation that rearranges the row list $(1, 1), (1, 2) \ldots, (1, n), (2, 1), (2, 2) \ldots (2, n), \ldots, (m, 1), (m, 2) \ldots (m, n)$ as $(1, 1), (2, 1) \ldots, (m, 1), (1, 2), (2, 2) \ldots (m, 2), \ldots, (1, n), (2, n) \ldots (m, n)$, etc.) In fact, $M(C \otimes D)N = D \otimes C$ for any C and D having the same sizes as A and B, respectively.

It is simple to verify that if digraphs G and H have adjacency matrices A and B, respectively, relative to vertex orderings $g_1, g_2, \ldots, g_m$ and $h_1, h_2, \ldots, h_n$, respectively, then $G \times H$ has adjacency matrix

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$A \otimes B$ relative to the ordering $(g_1, h_1), (g_1, h_2), \ldots, (g_1, h_n), (g_2, h_1), (g_2, h_2), \ldots, (g_2, h_n), \ldots, (g_m, h_1), (g_m, h_2), \ldots, (g_m, h_n)$ of its vertices.

Proposition 1 (*[2, Lemma 8.1.1]*). *Suppose G and H are digraphs with adjacency matrices A and B respectively. Then $G \cong H$ if and only if there is a permutation matrix P for which $PAP^{\mathrm{T}} = B$.*

In using Proposition 1, we keep in mind the connection between the matrix P and the corresponding isomorphism $\theta : G \to H$, namely $p_{ij} = 1$ if and only if $\theta(g_i) = h_j$. For example, suppose G and H are connected bipartite graphs, and the vertices of each graph are ordered so that all vertices of one partite set are listed first, followed by vertices in the other partite set. Since θ preserves partite sets, P must have block form $\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$ or $\begin{bmatrix} 0 & R \\ Q & 0 \end{bmatrix}$.

The following classic result of Lovász plays a major role in our main proof.

Proposition 2 (*Lovász, [7, Theorem 6]*). *Let A, B, C and D be digraphs, and suppose there exists a homomorphism $D \to C$. If $C \times A \cong C \times B$, then $D \times A \cong D \times B$.*

3. Results

We begin with some observations about the adjacency matrices of the direct product of two bipartite graphs. Suppose G and H are connected bipartite graphs with bipartitions $(G_0, G_1) = (\{w_1, w_2, \ldots w_m\}, \{x_1, x_2, \ldots, x_n\})$ and $(H_0, H_1) = (\{y_1, y_2, \ldots y_p\}, \{z_1, z_2, \ldots, z_q\})$, respectively. Then the adjacency matrices of G and H relative to these vertex orderings have the forms

$$\begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{bmatrix}, \tag{1}$$

respectively. Now, it is simple to check that the two components of $G \times H$ are induced on the vertex sets $(G_0 \times H_0) \cup (G_1 \times H_1)$ and $(G_0 \times H_1) \cup (G_1 \times H_0)$, respectively. We now construct adjacency matrices for the two components.

List the vertices of $(G_0 \times H_0) \cup (G_1 \times H_1)$ in the following order:

$$\begin{array}{llllllll}
(w_1, y_1), & (w_1, y_2), & \ldots, & (w_1, y_p), & (w_2, y_1), & (w_2, y_2), & \ldots, & (w_2, y_p), \\
\ldots & \ldots & \ldots & \ldots, & (w_m, y_1), & (w_m, y_2), & \ldots, & (w_m, y_p), \\
(x_1, z_1), & (x_1, z_2), & \ldots, & (x_1, z_q), & (x_2, z_1), & (x_2, z_2), & \ldots, & (x_2, z_q), \\
\ldots & \ldots & \ldots & \ldots, & (x_n, z_1), & (x_n, z_2), & \ldots, & (x_n, z_q).
\end{array}$$

Likewise, list the vertices of $(G_0 \times H_1) \cup (G_1 \times H_0)$ as

$$\begin{array}{llllllll}
(w_1, z_1), & (w_1, z_2), & \ldots, & (w_1, z_q), & (w_2, z_1), & (w_2, z_2), & \ldots, & (w_2, z_q), \\
\ldots & \ldots & \ldots & \ldots, & (w_m, z_1), & (w_m, z_2), & \ldots, & (w_m, z_q), \\
(x_1, y_1), & (x_1, y_2), & \ldots, & (x_1, y_p), & (x_2, y_1), & (x_2, y_2), & \ldots, & (x_2, y_p), \\
\ldots & \ldots & \ldots & \ldots, & (x_n, y_1), & (x_n, y_2), & \ldots, & (x_n, y_p).
\end{array}$$

As was observed in [1], it is simple to check that, relative to these vertex orderings, the two components of $G \times H$ have adjacency matrices

$$\begin{bmatrix} 0 & A \otimes B \\ A^{\mathrm{T}} \otimes B^{\mathrm{T}} & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & A \otimes B^{\mathrm{T}} \\ A^{\mathrm{T}} \otimes B & 0 \end{bmatrix}. \tag{2}$$

Now suppose the two components of $G \times H$ are isomorphic. Any isomorphism θ between them preserves their partite sets, so either $\theta(G_0 \times H_0) = G_0 \times H_1$ and $\theta(G_1 \times H_1) = G_1 \times H_0$, or $\theta(G_0 \times H_0) = G_1 \times H_0$ and $\theta(G_1 \times H_1) = G_0 \times H_1$. Proposition 1 (and the remark that follows it) applied to (2) guarantees permutation matrices Q and R for which either

$$\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} 0 & A \otimes B \\ A^{\mathrm{T}} \otimes B^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & A \otimes B^{\mathrm{T}} \\ A^{\mathrm{T}} \otimes B & 0 \end{bmatrix}$$

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or

$$\begin{bmatrix} 0 & R \\ Q & 0 \end{bmatrix}\begin{bmatrix} 0 & A\otimes B \\ A^{\mathrm{T}}\otimes B^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} 0 & R \\ Q & 0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & A\otimes B^{\mathrm{T}} \\ A^{\mathrm{T}}\otimes B & 0 \end{bmatrix}.$$

Multiplying the matrices, we see that either $Q(A\otimes B)R^{\mathrm{T}} = A\otimes B^{\mathrm{T}}$ in the first case, or $Q(A\otimes B)R^{\mathrm{T}} = A^{\mathrm{T}}\otimes B$ in the second. Renaming R as R^{T}, we have the following lemma.

Lemma 1. *Suppose connected bipartite graphs G and H have adjacency matrices $\begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{bmatrix}$, respectively. If the components of $G\times H$ are isomorphic, then there are permutation matrices Q and R for which either $Q(A\otimes B)R = A\otimes B^{\mathrm{T}}$ or $Q(A\otimes B)R = A^{\mathrm{T}}\otimes B$.*

Lemma 2. *Let bipartite graph G have matrix $\begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}$. If there are permutation matrices Q' and R' with $Q'AR' = A^{\mathrm{T}}$, then G has property π.*

Proof. Suppose $Q'AR' = A^{\mathrm{T}}$. Then

$$\begin{bmatrix} 0 & R'^{\mathrm{T}} \\ Q' & 0 \end{bmatrix}\begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} 0 & R'^{\mathrm{T}} \\ Q' & 0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}.$$

By Proposition 1 and the remark that follows it, we have an isomorphism from G to itself that reverses the partite sets, so G has property π. □

Now we can see how the proof of our main theorem will work. Suppose $G\times H$ has isomorphic components. By Lemma 1, there are permutation matrices Q and R for which either $Q(A\otimes B)R = A^{\mathrm{T}}\otimes B$ or $Q(A\otimes B)R = A\otimes B^{\mathrm{T}}$. Roughly speaking, we want to "cancel" the common factor of B (or A) and deduce that there are permutation matrices Q' and R' for which $Q'AR' = A^{\mathrm{T}}$ or $Q'BR' = B^{\mathrm{T}}$, from which Proposition 2 implies G or H has property π. The fact that this cancellation is justified follows from the next lemma.

Lemma 3. *Suppose A, A' and C are 0–1 matrices for which $C\neq O$, and A is square and has at least one nonzero entry in each row. Suppose also there are permutation matrices Q and R for which $Q(C\otimes A)R = C\otimes A'$. Then there are permutation matrices Q' and R' for which $Q'AR' = A'$. Also, if $Q(A\otimes C)R = A'\otimes C$, then there are permutation matrices Q' and R' for which $Q'AR' = A'$.*

Proof. We begin with the first statement. Suppose $Q(C\otimes A)R = C\otimes A'$. Let D be the digraph with adjacency matrix A, let D' be the digraph with adjacency matrix A', and let E be the digraph with adjacency matrix of block form

$$\begin{bmatrix} 0 & C \\ 0C^{\mathrm{T}} & 0 \end{bmatrix}. \tag{3}$$

(The lower left block is written as $0C^{\mathrm{T}}$ rather than 0 to indicate its size.) The adjacency matrices of $E\times D$ and $E\times D'$ are then

$$\begin{bmatrix} 0 & C\otimes A \\ 0 & 0 \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} 0 & C\otimes A' \\ 0 & 0 \end{bmatrix}$$

respectively. Notice that the lower left block of zeros in the matrix for $E\times D$ has the same size as $C^{\mathrm{T}}\otimes A$. Since A is square, this block is also the same size as $C^{\mathrm{T}}\otimes A^{\mathrm{T}} = (C\otimes A)^{\mathrm{T}}$, so this block can be multiplied by R^{T}. (Recall $Q(C\otimes A)R = C\otimes A'$, which means $R^{\mathrm{T}}(C\otimes A)^{\mathrm{T}}Q^{\mathrm{T}} = (C\otimes A')^{\mathrm{T}}$.) By letting Q and R^{T} be blocks in a larger permutation matrix, we have

$$\begin{bmatrix} Q & 0 \\ 0 & R^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} 0 & C\otimes A \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Q & 0 \\ 0 & R^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & Q(C\otimes A)R \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & C\otimes A' \\ 0 & 0 \end{bmatrix}.$$

Then from Proposition 1 we deduce $E\times D \cong E\times D'$.

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Recall that the graph E corresponds to the matrix C that we want to "eliminate". In pursuit of this goal, we next show that the E in the expression $E \times D \cong E \times D'$ can be replaced by a very simple graph K. Let K be the digraph on two vertices k and k' consisting of the single arc kk'. We claim that $K \times D \cong K \times D'$: Since $C \neq O$, we know from (3) that E has at least one arc ee'. Then the map sending k to e and k' to e' is a homomorphism $K \to E$. Proposition 2 now applies to give $K \times D \cong K \times D'$.

The adjacency matrices for $K \times D$ and $K \times D'$ are $\begin{bmatrix} 0 & A \\ 0A & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & A' \\ 0A' & 0 \end{bmatrix}$, respectively. Since $K \times D \cong K \times D'$, Proposition 1 guarantees a permutation matrix P for which

$$P \begin{bmatrix} 0 & A \\ 0A & 0 \end{bmatrix} P^{\mathrm{T}} = \begin{bmatrix} 0 & A' \\ 0A' & 0 \end{bmatrix}. \tag{4}$$

Since A has nonzero rows it follows that left-multiplying by P in (4) does not permute any of the rows of the upper blocks to rows of the lower blocks. Therefore P has block form $P = \begin{bmatrix} Q' & 0 \\ 0 & R' \end{bmatrix}$, so (4) yields $Q'AR'^{\mathrm{T}} = A'$. We now have permutation matrices Q' and R' with $Q'AR' = A'$. This proves the first part of the lemma.

For the second part of the lemma it must be shown that $Q(A \otimes C)R = A' \otimes C$ implies there are permutation matrices Q' and R' for which $Q'AR' = A'$. Let M and N be the permutation matrices for which $M(A \otimes C)N = C \otimes A$ and $M(A' \otimes C)N = C \otimes A'$. From $Q(A \otimes C)R = A' \otimes C$ it follows that $MQM^{\mathrm{T}}(C \otimes A)N^{\mathrm{T}}RN = C \otimes A'$. Then the first part of the lemma guarantees there are permutation matrices Q' and R' with $Q'AR' = A'$. □

Theorem 1. *Suppose G and H are connected bipartite graphs. The two components of $G \times H$ are isomorphic if and only if at least one of G or H has property π.*

Proof. As was noted earlier, the necessity was proved in [6]. We prove the sufficiency here. Suppose the two components of $G \times H$ are isomorphic. Using the notation established in (1), along with Lemma 1, we get that either $Q(A \otimes B)R = A^{\mathrm{T}} \otimes B$ or $Q(A \otimes B)R = A \otimes B^{\mathrm{T}}$.

If $Q(A \otimes B)R = A^{\mathrm{T}} \otimes B$, it follows that A is square; and since G is connected A has nonzero entries in each row and column. Thus Lemma 3 implies the existence of permutation matrices Q' and R' for which $Q'AR' = A^{\mathrm{T}}$, so Lemma 2 implies G has property π. On the other hand, if $Q(A \otimes B)R = A \otimes B^{\mathrm{T}}$, the same reasoning shows H has property π. □

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