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# Alternating series convergence: a visual proof

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## Abstract

**We describe a picture proof of the Alternating Series Test which uses simple comparisons of areas of rectangles to establish convergence.**

This article describes a short and compelling visual proof of the Alternating Series Test that uses comparisons of areas of rectangles to visualise and prove convergence and some related estimates. It can be quickly sketched on the board or given to a class as a handout.

The Alternating Series Test states that an alternating series

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots$$

of real numbers converges provided that $a_1 \geqslant a_2 \geqslant \cdots \geqslant 0$ and $\lim_{n\to\infty} a_n = 0$.

The standard calculus text proof is this: even partial sums $s_{2k} = a_1 - a_2 + a_3 - a_4 + \cdots - a_{2k}$ form an increasing sequence bounded above; odd partial sums $s_{2k+1} = a_1 - a_2 + a_3 - a_4 + \cdots - a_{2k} + a_{2k+1}$ form a decreasing sequence bounded below. Therefore, both partial sum sequences converge. Since $s_{2k+1} = s_{2k} + a_{2k+1}$ and $a_{2k+1} \to 0$. It follows that the alternating series itself converges to that number [see, e.g., (1)].

While this standard proof is short, it relies on two further abstractions: the monotone convergence theorem (to establish convergence of the even and odd partial sum sequences), and the fact that if the even and odd partial sums of a series both converge to the same number, then the series itself converges to that number. Our method avoids both of these by realising the partial sums and the limit as areas of rectangles. Here is the presentation we use with our students.

### Theorem (Alternating Series Test)

*An alternating series* $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 + \cdots$ *converges to a sum* $S$ *if* $a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 \geqslant \cdots \geqslant 0$ *and* $a_n \to 0$. *Moreover, for every* $k$, $s_{2k} < S < s_{2k+1}$ *and for every* $n$, $|S - s_n| < a_{n+1}$.

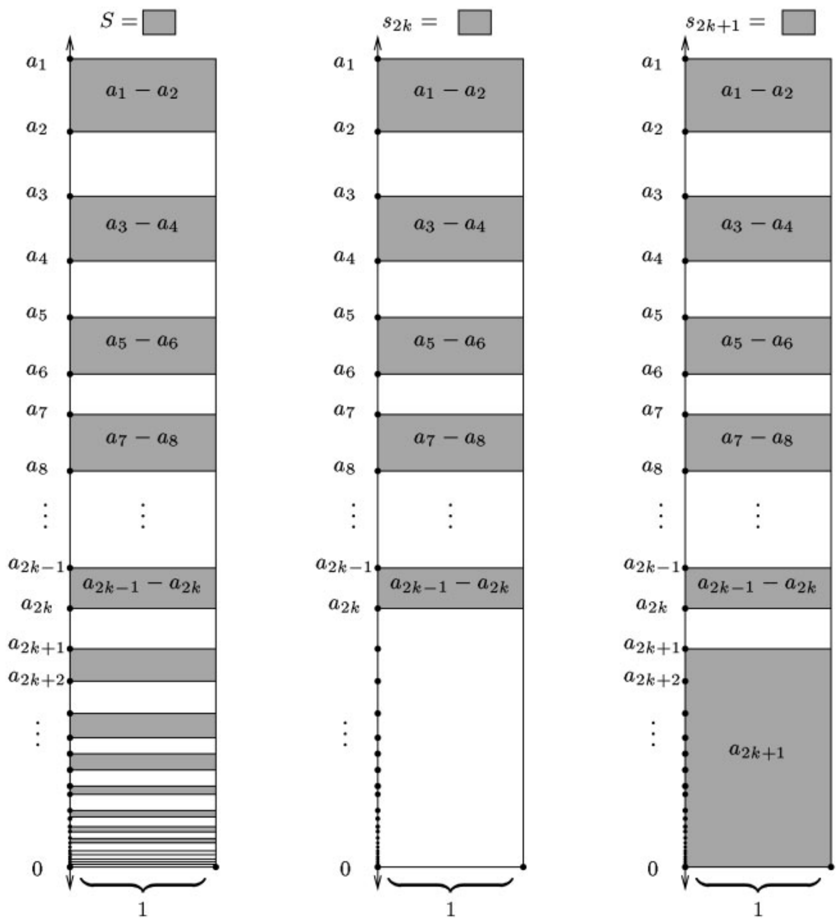


**Fig 1.** A visual proof of alternating series convergence.

**Proof:** We use the hypotheses of the theorem to draw Fig. 1, which shows the terms $a_1, a_2, a_3, \ldots$ of the alternating series as a decreasing sequence converging to zero along a vertical axis. Horizontal line segments one unit wide establish a strip of rectangles whose areas are the same as their vertical heights.

Let $S$ be the area of the shaded rectangles in the left-most tower (Fig. 1). This tower is followed by towers for even and odd partial sums $s_{2k}$ and $s_{2k+1}$. Comparison of these figures immediately establishes $s_{2k} < S < s_{2k+1}$.

Moreover, it is clear from Fig. 1 that the dark rectangles in the left-most tower below $a_{2k+1}$ represent the difference $S - s_{2k}$, and hence $|S - s_{2k}| < a_{2k+1}$. Also, the white rectangles below $a_{2k+2}$ represent the difference $s_{2k+1} - S$, hence $|S - s_{2k+1}| < a_{2k+2}$. Combining these inequalities gives $|S - s_n| < a_{n+1}$ for all $n$, as desired. As $a_n \to 0$, it further follows that $s_n \to S$, so the series indeed converges to $S$. □

We have had great success in teaching the Alternating Series Test this way. Students report that our approach is clearer than the standard textbook proof, which places the monotone convergence theorem and the subtleties of even and odd partial sum convergence as prerequisites. Using areas of rectangles replaces abstraction with concreteness and circumvents some difficulties.

However, we are not advocating that fine points be swept under the rug. Especially perceptive students notice that in defining $S$, we are tacitly using the monotone convergence theorem (as $S$ is the limit of the bounded increasing sequence of even partial sums). Such students may ask why the limits of even partial sums and of odd partial sums are not automatically the same thing. We are delighted when students themselves discover these subtleties. Our method provides a framework for understanding the convergence of alternating series which puts these deeper questions in their proper place: not as obstacles, but as avenues for discovery.

We invite interested teachers who would like to use our visual proof to download a copy from http://mas.lvc.edu/~lyons/pubs/altseriestest/proof.pdf.

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## References

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