Cyclicity of Graphs

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Abstract: The cyclicity of a graph is the largest integer n for which the graph is contractible to the cycle on n vertices. By analyzing the cycle space of a graph, we establish upper and lower bounds on cyclicity. These bounds facilitate the computation of cyclicity for several classes of graphs, including chordal graphs, complete n-partite graphs, n-cubes, products of trees and cycles, and planar graphs. J Graph Theory 32: 160–170, 1999

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1. INTRODUCTION

A graph G with vertex set $V(G)$ and edge set $E(G)$ is *contractible* to a graph H, if there is a partition $\{V_y | y \in V(H)\}$ of $V(G)$ with the property that each V_y induces a connected subgraph of G, and an edge of G joins V_x to V_y if and only if $xy \in E(H)$. Heuristically, H is obtained from G by collapsing to a vertex each subgraph induced by V_y and fusing any resulting multiple edges.

The *cyclicity*, $\eta(G)$, of a connected graph G is the largest integer n for which G is contractible to the n-cycle C_n. By convention, we set $C_1 = K_1$ and $C_2 = K_2$, so it is possible for a graph to have cyclicity 1 or 2. Clearly $\eta(C_n) = n$ for all natural numbers n, and it is easy to check that $\eta(G) \geq 3$ if and only if G has a cycle. Thus, $\eta(G) = 1$ if and only if $G = K_1$, and $\eta(G) = 2$ if and only if G is a nontrivial tree. Classification of graphs of cyclicity 3 (or greater) is an open question.

Little is known about the cyclicity of arbitrary graphs. Ideally, one would like to express this invariant in terms of some simple formula or structural property, but

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this may be difficult or impossible. Alternatively, one can concentrate on certain classes of graphs, and that is the approach we take in this article. In Section 2, we establish upper and lower bounds on cyclicity of an arbitrary connected graph. These bounds tell us the cyclicities of chordal graphs and complete n-partite graphs, and play a significant role in the remainder of the article. Section 3 deals with the cyclicities of Cartesian products of graphs. We show that the cyclicity of a product is bounded above and below by functions of certain invariants of its factors, and this leads to formulas for the cyclicity of products of cycles and trees (and, hence, also for n-cubes). The problem of computing cyclicity in the class of planar graphs is addressed in Section 4. It is proved there that the cyclicity of a two-connected planar graph equals the maximum number of disjoint paths joining two faces of a planar embedding. This suggests a polynomial-time algorithm for computing the cyclicity of such a graph.

Cyclicity was introduced by Blum in [3] as an aid in the study of a related invariant called *circularity* (see [1, 2]). Consequently, it is called *co-circularity* in [3]; we have renamed it here for clarity, and to emphasize that it is, by itself, an interesting concept. In this article, we investigate cyclicity topologically, in terms of contractions, whereas Blum uses combinatorial labelings. There is some overlap between the two approaches. The *cyclic* maps introduced below are equivalent to Blum's *co-admissible* maps, and our contractions to a cycle are similar to her k-*adequate subgraphs*.

We close this section with a summary of notation and definitions used throughout this article. Other definitions will be introduced as they arise.

If $X \subseteq V(G)$, then $G[X]$ denotes the subgraph of G induced by X. If K is a subgraph of G, then $G - K$ is the graph obtained by removing from G the vertices of K and all edges incident with them. If A and B are disjoint subgraphs of G, then $E(A, B) = \{vw \in E(G) | v \in V(A), w \in V(B)\}$. The cardinality of a set S is denoted $|S|$.

The vertices of the n-cycle C_n are identified with the elements of the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and written as $V(C_n) = \{0, 1, \ldots, n-1\}$, with $E(C_n) = \{a(a+1) | a \in V(C_n)\}$. As mentioned above, $C_1 = K_1$ and $C_2 = K_2$, though these are not (of course) regarded as cycles. The *length* of a cycle (or a path) is the cardinality of its edge set.

A *graph map* from a graph G to a graph H is a map $g : V(G) \to V(H)$ with the property that for each $vw \in E(G)$, either $g(v) = g(w)$ or $g(v)g(w) \in E(H)$. The graph map g is *surjective* if it is a surjective map on vertex sets and has the added property that every edge of H can be written as $g(v)g(w)$ for some $vw \in E(G)$. A graph map g is often denoted as $g : G \to H$. If $y \in V(H)$, then $G[g^{-1}(y)]$ is called the *fiber of g over y*.

The *edge space* $\mathcal{E}(G)$ of a graph G is the power set of $E(G)$ endowed with the structure of a vector space over the two-element field $\mathbb{F}_2 = \{0, 1\}$. Addition in $\mathcal{E}(G)$ is symmetric difference of sets, and zero is the empty set. Similarly, the *vertex space* $\mathcal{V}(G)$ of G is the power set of $V(G)$ viewed as a vector space over $\mathbb{F}_2$. The set $E(G)$ is a basis for $\mathcal{E}(G)$, and $V(G)$ is a basis for $\mathcal{V}(G)$. A graph map

$g : G \to H$ induces a linear map $g^* : \mathcal{E}(G) \to \mathcal{E}(H)$ defined on the basis $E(G)$ as $g^*(vw) = 0$ if $g(v) = g(w)$, and $g^*(vw) = g(v)g(w)$ if $g(v)g(w) \in E(H)$. Likewise g induces a linear map $g' : \mathcal{V}(G) \to \mathcal{V}(H)$ defined on the basis $V(G)$ as $g'(v) = g(v)$.

For any graph G, there is a linear *incidence map* $B_G : \mathcal{E}(G) \to \mathcal{V}(G)$ whose effect on the basis $E(G)$ is $B_G(vw) = v + w$. The kernel of this map is denoted $\mathcal{C}(G)$ and called the *cycle space* of G, because it is the subspace of $\mathcal{E}(G)$ spanned by the edge sets of cycles in G ([6], Proposition 1.9.7). The cycle space consists exactly of the edge sets $E(K)$ of subgraphs K of G, which have no vertex of odd degree ([6], Proposition 1.9.2). If G is connected, $\mathcal{C}(G)$ has dimension $|E(G)| - |V(G)| + 1$ ([6], Theorem 1.9.6). A basis for $\mathcal{C}(G)$, which consists entirely of edge sets of cycles is called a *cycle basis*. Since the induced cycles of G span $\mathcal{C}(G)$ ([6], Proposition 1.9.1), it is always possible to find a cycle basis consisting of induced cycles.

To avoid proliferation of notation, we will often blur the distinction between a subgraph K of G and its edge set $E(K) \in \mathcal{E}(G)$. Thus, if J and K are subgraphs, an expression such as $J + K$ always means $E(J) + E(K)$, with the operation taking place in $\mathcal{E}(G)$ (or, more often, in $\mathcal{C}(G)$).

A graph map $g : G \to H$ gives rise to the following diagram, whose right-hand square is commutative (check this on the basis $E(G)$).

$$
\begin{array}{ccccc}
\mathcal{C}(G) & \hookrightarrow & \mathcal{E}(G) & \xrightarrow{B_G} & \mathcal{V}(G) \\
 & & \downarrow g^* & & \downarrow g' \\
\mathcal{C}(H) & \hookrightarrow & \mathcal{E}(H) & \xrightarrow{B_H} & \mathcal{V}(H).
\end{array}
$$

If $K \in \mathcal{C}(G)$, then $B_H(g^*(K)) = g'(B_G(K)) = g'(0) = 0$, so $g^*(K) \in \ker(B_H) = \mathcal{C}(H)$. It follows that g^* restricts to a map $g^* : \mathcal{C}(G) \to \mathcal{C}(H)$.

The map $g^* : \mathcal{C}(G) \to \mathcal{C}(H)$ induced by a graph map $g : G \to H$ plays a key role in this article. Of particular interest will be the case $H = C_n$, in which $\mathcal{C}(C_n) = \{\emptyset, E(C_n)\}$ is isomorphic to $\mathbb{F}_2$.

2. UPPER AND LOWER BOUNDS ON CYCLICITY

If G is contractible to C_n, then there is a partition $\{V_a | a \in V(C_n) = \mathbb{Z}_n\}$ of $V(G)$ with each $G[V_a]$ connected, and, for $a \neq b, E(V_a, V_b) \neq \emptyset$ exactly when $ab \in E(C_n)$. This gives rise to a graph map $g : G \to C_n$ defined by $g(v) = a$ if $v \in V_a$. This map is clearly surjective, and each fiber $G[g^{-1}(a)] = G[V_a]$ is connected. With this in mind, we call a graph map $g : G \to C_n$ *cyclic* if it is surjective and the fiber $G[g^{-1}(a)]$ is connected for each $a \in V(C_n)$. Thus, each contraction of G to C_n gives rise to a cyclic map $g : G \to C_n$. On the other hand, given a cyclic map $g : G \to C_n$, the graph G is contractible to C_n, for $\{g^{-1}(a) | a \in V(C_n)\}$ is a partition of $V(G)$ with each $G[g^{-1}(a)]$ connected and $E(g^{-1}(a), g^{-1}(b)) \neq \emptyset$ exactly when $ab \in E(C_n)$. Therefore, we have the following.

Proposition 2.1. The cyclicity of a graph G is the largest integer n for which there is a cyclic map $G \to C_n$.

In practice, this proposition is used to establish lower bounds on the cyclicity of a graph; if we can exhibit a cyclic map $G \to C_n$, then $n \leq \eta(G)$. Before proving a result on upper bounds, we will need a lemma.

Lemma 2.1. If $n \geq 3$ and $g : G \to C_n$ is cyclic, then $g^* : \mathcal{C}(G) \to \mathcal{C}(C_n)$ is nonzero.

Proof. Since g is surjective, for each $a(a+1) \in E(C_n)$ there is an edge $v_a w_{a+1} \in E(G)$ with $g(v_a) = a$ and $g(w_{a+1}) = a+1$. Since the fibers of g are connected, for each $a \in V(C_n)$, there is a w_a–v_a path P_a with $g(V(P_a)) = a$. Form the cycle $C = P_0 v_0 w_1 P_1 v_1 w_2 P_2 v_2 w_3 \cdots P_{n-1} v_{n-1} w_0 \in \mathcal{C}(G)$, and notice that $g^*(E(C)) = \sum_{a \in V(C_n)} (g^*(E(P_a)) + g^*(v_a w_{a+1})) = \sum_{a \in V(C_n)} (0 + a(a+1)) = E(C_n) \neq 0$. (The condition $n \geq 3$ implies each $g^*(v_a w_{a+1}) = a(a+1)$ is distinct, so there is no cancellation of the terms $a(a+1)$ in the sum.) ∎

Proposition 2.2. The cyclicity of a connected graph G, which contains a cycle, is at most the length of the longest cycle in any cycle basis of $\mathcal{C}(G)$.

Proof. By Proposition 2.1, there is a cyclic map $g : G \to C_{\eta(G)}$, and $g^* : \mathcal{C}(G) \to \mathcal{C}(C_{\eta(G)})$ is nonzero by Lemma 2.1. Therefore, any cycle basis $\mathcal{B}$ of $\mathcal{C}(G)$ has an element $B \in \mathcal{B}$ for which $g^*(B) \neq 0$, meaning $g^*(B) = E(C_{\eta(G)})$. It follows that $\eta(G)$ is at most the number of edges in B. ∎

Later we will obtain sharp upper bounds for cyclicity by applying Proposition 2.2 to a judicious choice of cycle basis. For now we mention an immediate corollary and two of its consequences. Since $\mathcal{C}(G)$ is spanned by the induced cycles in G, we have the following.

Corollary 2.1. The cyclicity of a graph that contains a cycle is at most the length of its longest induced cycle.

This corollary tells us the cyclicity of the complete graphs, though it applies to the wider class of chordal graphs. Recall that a graph is *chordal* if all its induced cycles are triangles.

Corollary 2.2. Every chordal graph that contains a cycle has cyclicity 3. In particular, $\eta(K_n) = 3$ for $n \geq 3$.

To conclude this section, we characterize the cyclicity of the complete n-partite graphs $K(p_1, \ldots, p_n)$.

Proposition 2.3. If $p \geq 1$, then $\eta(K(1,p)) = 2$; also $\eta(K(2,2)) = 4$. All other complete n-partite graphs ($n \geq 2$) have cyclicity 3.

Proof. Since $K(1,p)$ is a nontrivial tree, its cyclicity is 2, and $\eta(K(2,2)) = 4$ because $K(2,2) \cong C_4$. Let $G = K(p_1, p_2, \ldots, p_n)$ be any other complete n-partite graph. One easily checks that G has a cycle, so $3 \leq \eta(G)$. Combine this with the fact that all induced cycles of G have length 3 or 4, and Corollary 2.1 implies $3 \leq \eta(G) \leq 4$. To finish the proof we show that $G = K(2,2)$ if $\eta(G) = 4$.

Suppose that $\eta(G) = 4$, so G is contractible to C_4. Then there is a partition $\{V_0, V_1, V_2, V_3\}$ of $V(G)$ (indexed over $V(C_4) = \mathbb{Z}_4$) with each $G[V_a]$ connected and, for $a \neq b, E(V_a, V_b) \neq \emptyset$ if and only if $b = a + 1$ or $a = b + 1$. To see that $G = K(2, 2) = C_4$, we just need to show that each V_a consists of a single vertex. Take any $a \in V(C_4)$ and notice that $E(V_a, V_{a+2}) = \emptyset$, so the vertices $V_a \cup V_{a+2}$ belong to the same partite set of G. Then $G[V_a]$ has no edges, yet is connected, so it is a single vertex. ∎

3. PRODUCTS

Recall that the *Cartesian product* of graphs G and H is the graph $G \times H$ with $V(G \times H) = V(G) \times V(H)$ and $(v, x)(w, y) \in E(G \times H)$, if $v = w$ and $xy \in E(H)$, or $x = y$ and $vw \in E(G)$. If L and M are subgraphs of G and H, respectively, then $L \times M$ is regarded as a subgraph of $G \times H$ in the obvious manner. Given graphs $G_1, \ldots, G_n$, we write $G_1 \times \cdots \times G_n = \prod_{i=1}^{n} G_i$.

In this section, we relate the cyclicity of a Cartesian product of graphs to the cycle structures of its factors. Though our main result gives upper and lower bounds for the cyclicity of a product, this is sometimes enough to yield an exact value. For instance, we obtain formulas for the cyclicities of products of cycles and trees, hence also for n-cubes.

If $\mathcal{B}$ is a cycle basis for $\mathcal{C}(G)$, let $l(\mathcal{B})$ be the length of the longest cycle in $\mathcal{B}$. Define $\lambda(G) = \min\{l(\mathcal{B}) | \mathcal{B}$ is a cycle basis for $\mathcal{C}(G)\}$, or $\lambda(G) = 0$ if G has no cycles. Our primary result in this section is the following.

Proposition 3.1. If G and H are nontrivial connected graphs, then $\max\{4, \eta(G), \eta(H)\} \leq \eta(G \times H) \leq \max\{4, \lambda(G), \lambda(H)\}$.

Proof. To prove that $\max\{4, \eta(G), \eta(H)\} \leq \eta(G \times H)$, we show that $\eta(G) \leq \eta(G \times H), \eta(H) \leq \eta(G \times H)$, and $4 \leq \eta(G \times H)$. Choose a cyclic map $g : G \to C_{\eta(G)}$, and let $\pi : V(G \times H) \to V(G)$ be projection to the first factor, which is a surjective graph map. Then the composition $g \circ \pi : G \times H \to C_{\eta(G)}$ is a surjective graph map. Moreover, this composition is cyclic, for given any $a \in V(C_{\eta(G)})$, the fiber $(G \times H)[(g \circ \pi)^{-1}(a)] = (G \times H)[g^{-1}(a) \times V(H)] = G[g^{-1}(a)] \times H$ is connected. By Proposition 2.1, it follows that $\eta(G) \leq \eta(G \times H)$, and similarly $\eta(H) \leq \eta(G \times H)$.

To show that $4 \leq \eta(G \times H)$, we describe a contraction of $G \times H$ to C_4. Choose vertices $v_0 \in V(G)$ and $y_0 \in V(H)$ for which both $G' = G - v_0$ and $H' = H - y_0$ are connected. Using indices from $V(C_4) = \mathbb{Z}_4$, form a partition $\{V_0, V_1, V_2, V_3\}$ of $V(G \times H)$ as follows: $V_0 = \{(v_0, y_0)\}, V_1 = V(G' \times y_0), V_2 = V(G' \times H')$ and $V_3 = V(v_0 \times H')$. Each $(G \times H)[V_a]$ is connected, and $E(V_a, V_b) \neq \emptyset$ exactly when $ab \in E(C_4)$, so $G \times H$ is contractible to C_4. This completes the proof that $\max\{4, \eta(G), \eta(H)\} \leq \eta(G \times H)$.

It remains to show $\eta(G \times H) \leq \max\{4, \lambda(G), \lambda(H)\}$. Since $\eta(G \times H) \leq \lambda(G \times H)$ by Proposition 2.2, it suffices to show that $\lambda(G \times H) \leq \max\{4, \lambda(G),$

$\lambda(H)\}$, and this can be accomplished by producing a cycle basis of $\mathcal{C}(G \times H)$ whose longest element has length $\max\{4, \lambda(G), \lambda(H)\}$. Construction of this basis takes up the remainder of the proof.

Take cycle bases $\mathcal{B}_G$ and $\mathcal{B}_H$ of $\mathcal{C}(G)$ and $\mathcal{C}(H)$ whose longest cycles have lengths $\lambda(G)$ and $\lambda(H)$, respectively. (If G has no cycles, then $\mathcal{B}_G = \emptyset$, and similarly for H.) Let S be a spanning tree of G and let T be a spanning tree of H. Form the following sets of cycles in $\mathcal{C}(G \times H)$:

$$\mathcal{F} = \{e \times f \,|\, e \in E(S), f \in E(T)\},$$

$$\mathcal{G} = \{C \times y \,|\, C \in \mathcal{B}_G, y \in V(H)\},$$

$$\mathcal{H} = \{v \times C \,|\, v \in V(G), C \in \mathcal{B}_H\}.$$

The longest cycle in $\mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$ has length $\max\{4, \lambda(G), \lambda(H)\}$, by construction. The proof will be complete when we show that this set is a basis for $\mathcal{C}(G \times H)$.

Now, $\mathcal{G} \cup \mathcal{H}$ is an independent set, for $\mathcal{G}$ and $\mathcal{H}$ are, respectively, bases for the cycle spaces of the edge-disjoint subgraphs $G \times V(H)$ and $V(G) \times H$ of $G \times H$. We next show that $\mathcal{F}$ is independent, then that $\mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$ is independent.

Suppose that $0 = \sum_{e \in E(S)} \sum_{f \in E(T)} \alpha_{ef}(e \times f)$, where each α_{ef} is in $\mathbb{F}_2$. Take any $e_0 \in E(S)$ and $f_0 = x_0 y_0 \in E(T)$. We prove that $\mathcal{F}$ is an independent set by showing $\alpha_{e_0 f_0} = 0$. Rewrite the above equation as

$$\sum_{e \in E(S)} \alpha_{ef_0}(e \times f_0) = \sum_{e \in E(S)} \left(\sum_{f \in E(T) - \{f_0\}} \alpha_{ef}(e \times f) \right).$$

Think of the left-hand side as an edge set, and notice that it contains no edges of the form $v \times f_0, v \in V(G)$, for no such edge appears in the right-hand sum. The left-hand side is, therefore, an element of $\mathcal{C}(G \times H)$ consisting entirely of edges in the forest $S \times \{x_0, y_0\}$, so it must be zero (if every vertex of a forest has even degree, the forest has no edges). Thus, $0 = \sum_{e \in E(S)} \alpha_{ef_0}(e \times f_0)$, or rather $\alpha_{e_0 f_0}(e_0 \times f_0) = \sum_{e \in E(S) - \{e_0\}} \alpha_{ef_0}(e \times f_0)$. The cycle $e_0 \times f_0$ on the left contains the edge $e_0 \times x_0$, but this edge does not appear on the right. Consequently, $\alpha_{e_0 f_0} = 0$, and this completes the proof that $\mathcal{F}$ is a set of independent vectors.

So far we have that $\mathcal{F}$ and $\mathcal{G} \cup \mathcal{H}$ are independent sets and we wish to show that $\mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$ is independent. To do this, suppose that A and B are in the spans of $\mathcal{F}$ and $\mathcal{G} \cup \mathcal{H}$, respectively, and $A + B = 0$. We show that B (hence, also A) is zero. If B were nonzero, then–-by definition of $\mathcal{G}$ and $\mathcal{H}$–-some of its edges would form a cycle C in one of the subgraphs $G \times y$ or $v \times H$ for $y \in V(H), v \in V(G)$. Without loss of generality, let's say that C is a cycle in $G \times y$. Since $A = B$ (because $A + B = 0$), it follows that $E(C) \subseteq B \cap E(G \times y) = A \cap E(G \times y) \subseteq E(S \times y)$. This means that the edges of the cycle C lie in the tree $S \times y$. Since this is impossible, we conclude $B = 0$.

Finally, we check that the independent set $\mathcal{F} \cup G \cup \mathcal{H}$ is actually a basis for $\mathcal{C}(G \times H)$ by showing the cardinality of the former equals the dimension of the latter. Say that G has p vertices and q edges, while H has r vertices and s edges.

Then $|\mathcal{F} \cup \mathcal{G} \cup \mathcal{H}| = |\mathcal{F}| + |\mathcal{G}| + |\mathcal{H}| = (p-1)(r-1) + (q-p+1)r + p(s-r+1) = (ps+qr)-pr+1 = |E(G\times H)|-|V(G\times H)|+1 = \dim(\mathcal{C}(G\times H))$. ∎

In the above proof, we showed that $\max\{4, \eta(G), \eta(H)\} \leq \eta(G \times H) \leq \lambda(G\times H) \leq \max\{4, \lambda(G), \lambda(H)\}$. Combining these inequalities inductively gives the following immediate generalization.

Proposition 3.2. If $G_1, \ldots, G_n$ is a set of connected nontrivial graphs, then $\max\{4, \eta(G_1), \ldots, \eta(G_n)\} \leq \eta(\prod_{i=1}^{n} G_i) \leq \max\{4, \lambda(G_1), \ldots, \lambda(G_n)\}$.

It sometimes happens that a graph G obeys $\eta(G) \geq \lambda(G)$. For example, $\eta(G) = \lambda(G)$ if G is a cycle, and $\eta(G) > \lambda(G)$ if G is a tree. When this is the case for all factors in a product, the upper and lower bounds in Proposition 3.2 coincide.

Proposition 3.3. If each of the nontrivial graphs $G_1, \ldots, G_n$ satisfies $\eta(G) \geq \lambda(G)$, then $\eta(\prod_{i=1}^{n} G_i) = \max\{4, \eta(G_1), \ldots, \eta(G_n)\}$.

Corollary 3.1. Given integers $k_1,\ k_2, \ldots, k_n \geq 2$, then $\eta(\prod_{i=1}^{n} C_{k_i}) = \max\{4, k_1, k_2, \ldots, k_n\}$.

The cyclicity of the n-cube $Q_n = \prod_{i=1}^{n} K_2$ can now be easily computed.

Corollary 3.2. The cyclicity of any product of nontrivial trees is 4. In particular, for $n \geq 2$, the cyclicity of the n-cube is 4.

4. PLANAR GRAPHS

In this section, we prove that the cyclicity of a two-connected planar graph equals the maximum number of disjoint paths joining two faces of an embedding of the graph in the plane. From this result comes a polynomial-time algorithm, which computes the cyclicity of such a graph.

A planar graph with a fixed embedding in the Euclidean plane $\mathbb{R}^2$ is called a *plane* graph and is regarded as a subspace of $\mathbb{R}^2$. If $G \subseteq \mathbb{R}^2$ is a plane graph, then the connected components of $\mathbb{R}^2 - G$ are called the *faces* of G, and the set of faces is denoted by $F(G)$. The *boundary* ∂Y of a face Y is the subgraph of G, which lies in the topological closure of Y. If G is a two-connected plane graph, then the boundary of each of its faces is a cycle ([6], Proposition 4.2.5).

Suppose that G is a plane graph and $g : G \to C_n$ is cyclic. A face Y of G is *saturated*, if the restriction $g : \partial Y \to C_n$ is surjective. Clearly, Y is saturated if $g^*(\partial Y) = C_n$. We will need the following lemma, due to Blum ([3], Theorem 4.6). We offer a different proof.

Lemma 4.1. If G is a two-connected plane graph and $g : G \to C_n$ $(n \geq 3)$ is cyclic, then G has two saturated faces.[1]

Proof. Let G and g be as in the statement of the lemma. Since G is two-connected, ∂Y is a cycle for each $Y \in F(G)$. By Theorem 5.4 of [4], the set

[1] Blum proves a stronger result: If $n \geq 3$, there are *exactly* two saturated faces.

$\mathcal{Y} = \{\partial Y | Y \in F(G)\}$ spans $\mathcal{C}(G)$. (Alternatively, just observe that any cycle C of G divides $\mathbb{R}^2$ into two regions, and C equals the sum of the boundaries of all the faces in one of these regions). By Lemma 2.1, the map $g^* : \mathcal{C}(G) \to \mathcal{C}(C_n) = \{0, C_n\}$ is nonzero, so it is nonzero on some element of the spanning set $\mathcal{Y}$. Thus, there is a face $Z \in F(G)$ for which $g^*(\partial Z) = C_n$, meaning that Z is saturated. Now, each edge of G is in the boundary of exactly two faces, so $0 = \sum_{Y \in F(G)} \partial Y$, or $\partial Z = \sum_{Y \in F(G)-\{Z\}} \partial Y$. Taking g^* of both sides, $C_n = \sum_{Y \in F(G)-\{Z\}} g^*(\partial Y)$, so it is impossible that $g^*(\partial Y) = 0$ for every $Y \in F(G) - \{Z\}$. Hence, there must be some $Y \in F(G) - \{Z\}$ for which $g^*(\partial Y) \neq 0$, so $g^*(\partial Y) = C_n$. Therefore, Y and Z are saturated. ∎

As Blum observed ([3], Corollary 4.7), this gives rise to an upper bound for the cyclicity of a two-connected plane graph G; namely, $\eta(G)$ is bounded above by the maximum number n for which G has two faces with at least n vertices. There are, however, cases in which this bound is not attained. For example, any planar embedding of the graph $K(2, 3)$ has exactly three faces, whose boundaries are 4-cycles, yet its cyclicity is 3 by Proposition 2.3. In the next proposition, we characterize cyclicity of a two-connected plane graph in terms of the maximum number of disjoint paths joining two faces.

We say there are n disjoint paths joining faces Y and Z of the plane graph G, if G contains n paths with pairwise disjoint vertex sets, each joining a vertex of ∂Y to a vertex of ∂Z. (If ∂Y and ∂Z share a vertex, it is possible that such a path consists only of that vertex.) Given two faces Y and Z, let $M(Y, Z)$ denote the maximum number of disjoint paths joining Y and Z.

Proposition 4.1. If G is a two-connected plane graph, then $\eta(G) = \max\{M(Y, Z) | Y, Z \in F(G)\}$.

Proof. Let G be a two-connected plane graph, and choose a cyclic map $g : G \to C_{\eta(G)}$. Now, $\eta(G) \geq 3$ because G, being two-connected, contains a cycle. By Lemma 4.1, G has two saturated faces Y_0 and Z_0, so, for each $a \in V(C_{\eta(G)})$, the fiber $G[g^{-1}(a)]$ contains vertices of both Y_0 and Z_0. Since the fibers are connected and pairwise disjoint, there are $\eta(G)$ disjoint paths joining Y_0 to Z_0. Consequently, $\eta(G) \leq M(Y_0, Z_0) \leq \max\{M(Y, Z) | Y, Z \in F(G)\}$.

To establish the reverse inequality, choose any two faces Y and Z of G. It is enough to exhibit a cyclic map $g : G \to C_{M(Y,Z)}$, for then $\eta(G) \geq M(Y, Z)$ (by Proposition 2.1), so $\eta(G) \geq \max\{M(Y, Z) | Y, Z \in F(G)\}$.

To define g, set $n = M(Y, Z)$ and choose n disjoint paths $P_0, P_1, \ldots, P_{n-1}$ joining Y to Z. Assume the paths are listed in clockwise order around the face Y, and that the indexing is taken over $V(C_n) = \mathbb{Z}_n$. Let $K = G - \bigcup_{a \in \mathbb{Z}_n} P_a$, and observe that each component of K is adjacent to exactly one path P_a or exactly two paths P_a and P_{a+1} (for some $a \in \mathbb{Z}_n$). For each $a \in \mathbb{Z}_n$, let L_a be the union (possibly empty) of the components of K that are adjacent only to P_a. Let M_a be the union of the components of K that are adjacent to both P_a and P_{a+1}. Then $\{V(P_a) \cup V(L_a) \cup V(M_a) | a \in \mathbb{Z}_n\}$ is a partition of $V(G)$. Define

$g : V(G) \to V(C_n)$ as $g(v) = a$ if $v \in V(P_a) \cup V(L_a) \cup V(M_a)$. The rest of the proof is devoted to proving g is cyclic.

To confirm that g is a graph map, notice that any edge $e = vw$ of G is in one of the sets $E(P_a), E(L_a), E(M_a), E(L_a, P_a), E(M_a, P_a), E(M_a, P_{a+1})$, or $E(P_a, P_{a+1})$ for some $a \in \mathbb{Z}_n$. It follows that either $g(v) = g(w)$, or $g(v)g(w) = a(a+1) \in E(C_n)$. Further, for each $a \in \mathbb{Z}_n$, the fiber $G[g^{-1}(a)] = G[V(P_a) \cup V(L_a) \cup V(M_a)]$ is connected, because each of the connected components of L_a and M_a is adjacent to the (connected) path P_a. Finally, g is surjective, for given an edge $a(a+1) \in E(C_n)$, the cycle ∂Y clearly contains edges in $E(M_a, P_{a+1})$ or $E(P_a, P_{a+1})$. If vw is such an edge, then $g(v)g(w) = a(a+1)$. ∎

The hypothesis of two-connectedness in Proposition 4.1 is somewhat artificial in that it is imposed only to ensure that the boundary of each face is a cycle (which simplifies the proof). The cyclicity of an arbitrary plane graph can be found by applying the proposition to the two-connected blocks of the graph, as will be implied by Proposition 5.1, below.

We close this section by describing an algorithm—arising from Proposition 4.2—that computes the cyclicity of a two-connected plane graph. Given faces $Y, Z \in F(G)$, let G_{YZ} be a new graph formed from G by adding two new vertices y and z and new edges joining y and z to the vertices of ∂Y and ∂Z, respectively. Specifically, $V(G_{YZ}) = V(G) \cup \{y, z\}$ and $E(G_{YZ}) = E(G) \cup \{yx | x \in V(\partial Y)\} \cup \{zx | x \in V(\partial Z)\}$. Now $M(Y, Z)$ can be interpreted as the maximum number of internally disjoint y-z paths in G_{YZ}. This number is the value of the maximal flow in a certain network associated to G_{YZ}, and this flow can be computed using, say, the Edmunds–Karp max-flow min-cut algorithm ([5], Algorithm 5.1). For details, the reader is referred to Theorem 5.9 of [5], as well as Section 5.4, where it is shown that the complexity of using this method to compute $M(Y, Z)$ is $O(pq^2)$, where G has p vertices and q edges.

Proposition 4.1 says $\eta(G)$ can be found by computing $M(Y, Z)$ for all pairs of faces in G and selecting the largest value thus obtained. If G has r faces, then $M(Y, Z)$ must be computed $r(r-1)/2$ times. But $r(r-1)/2 < r^2 = (q-p+2)^2 < q^2$, so the complexity of using this method to find $\eta(G)$ is $O(pq^4)$.

5. CONCLUSION

There are many open questions concerning cyclicity. Foremost is the determination of the cyclicity of an arbitrary graph.

Problem A. Express the cyclicity of an arbitrary connected graph in terms of some structural property.

Since we have a polynomial-time algorithm for determining the cyclicity of two-connected planar graphs, it is natural to ask if one exists for arbitrary graphs.

Problem B. Find a good algorithm that computes cyclicity of an arbitrary connected graph, or show this problem is NP-hard.

These may be difficult problems. Alternatively, one could concentrate on certain classes of graphs, such as bipartite graphs, cubic graphs, cages, circulants, etc. In another direction, one could ask what graphs have a given cyclicity. Since we understand which graphs have cyclicity 1 or 2, a natural problem is the following.

Problem C. Characterize the graphs of cyclicity 3 (or greater).

In answering these and other questions, it may be useful to impose additional structure on our graphs. We close with one final result which implies that—as far as computation of cyclicity is concerned—there is no loss of generality in assuming a graph is irreducible.

Recall that a graph is *irreducible* if it is not separated by any complete subgraph. If G is not irreducible, it can be decomposed into two induced subgraphs G_1 and G_2 for which $G = G_1 \cup G_2$ and $K = G_1 \cap G_2$ is a complete graph. In turn, each of G_1 and G_2 may be similarly decomposed, and so on, until all subgraphs obtained are irreducible. These subgraphs are called *irreducible components* of G. Though G can always be decomposed into irreducible components, this decomposition is generally not unique ([6], Chapter 12, Exercise 16). The next proposition shows that the computation of the cyclicity of a graph can be reduced to the computation of the cyclicities of its irreducible components. This proposition is a generalization of Theorem 3.6 of [2], applied to cyclicity (rather than circularity).

Proposition 5.1. If the connected graph G can be decomposed into irreducible components $G_1, \ldots, G_n$, then $\eta(G) = \max\{\eta(G_1), \ldots, \eta(G_n)\}$.

Proof. It suffices to show that $\eta(G) = \max\{\eta(G_1), \eta\ (G_2)\}$, when $G = G_1 \cup G_2$ and $K = G_1 \cap G_2$ is a complete separating subgraph of G.

Let $g : G \to C_n$ be a cyclic map. According to Proposition 2.1, to prove $\eta(G) \le \max\{\eta(G_1), \eta(G_2)\}$, it is enough to show that g restricts to a cyclic map on one of G_1 or G_2. Certainly the restriction of g to either G_1 or G_2 inherits from g the property of being a graph map. Also, note that the restriction $g : G_1 \to C_n$ (and $g : G_2 \to C_n$) has connected fibers: If $v, w \in V(G_1)$ are in the fiber over $a \in C_n$, then, since $g : G \to C_n$ has connected fibers, there is a v-w path P in G with $g(V(P)) = a$. Let P be the shortest path with this property. Then P lies entirely in G_1, for otherwise $P = v_1 v_2 \cdots v_i \cdots v_j \cdots v_k \cdots v_{l-1} v_l$, with $v_1 = v, v_l = w$, $v_j \in V(G_2) - V(G_1)$, and $v_i, v_k \in V(K)$; then $P' = v_1 v_2 \cdots v_i v_k \cdots v_{l-1} v_l$ is a shorter v-w path with $g(V(P')) = a$. Thus, $g : G_1 \to C_n$ (similarly, $g : G_2 \to C_n$) has connected fibers. So far we have shown that each restriction $g : G_1 \to C_n$ and $g : G_2 \to C_n$ is a graph map with connected fibers, and it remains to confirm that one of these restrictions is surjective. Let $\mathcal{B}$ be a basis of $\mathcal{C}(G)$ consisting of induced cycles, so every element of $\mathcal{B}$ is a cycle in G_1 or G_2. Now, $g^* : \mathcal{C}(G) \to \mathcal{C}(C_n)$ is nonzero by Lemma 2.1, so $g^*(B) = C_n$ for some $B \in \mathcal{B}$. Since B lies entirely in one of G_1 or G_2, it follows that one of the restrictions of g to G_1 or G_2 is surjective. This completes the proof that $\eta(G) \le \max\{\eta(G_1), \eta(G_2)\}$.

To prove $\eta(G) \ge \max\{\eta(G_1), \eta(G_2)\}$, we show that any cyclic map $g : G_1 \to C_n$ (or $g : G_2 \to C_n$) extends to a cyclic map $\tilde{g} : G \to C_n$. Suppose that $g : G_1 \to C_n$ is cyclic. Let $W_1, W_2, \ldots, W_k$ be the components of $G_2 - G_1$, and, for each $i \in \{1, 2, \ldots, k\}$, choose a vertex $w_i \in V(K) \subseteq V(G_1)$ that is adjacent to W_i. Define $\tilde{g}(v) = g(v)$ if $v \in V(G_1)$ and $\tilde{g}(v) = g(w_i)$ if $v \in V(W_i)$. Notice that $\tilde{g}$ has been defined in such a way that each of its fibers is connected. Moreover, it inherits surjectivity from g. We just need to check that it is indeed a graph map. To do this, take $vw \in E(G)$. We must show that either $\tilde{g}(v) = \tilde{g}(w)$ or $\tilde{g}(v)\tilde{g}(w) \in E(C_n)$. Now, either $vw \in E(G_1)$, or $vw \in E(W_i)$, or $vw \in E(G_1, W_i)$ for some i. In the first case, $\tilde{g}(v) = \tilde{g}(w)$ or $\tilde{g}(v)\tilde{g}(w) \in E(C_n)$, because $\tilde{g}$ restricts to the graph map g on G_1. In the second case, $\tilde{g}(v) = \tilde{g}(w)$ by definition. Finally, if $vw \in E(G_1, W_i)$, then $v \in V(K)$ and $w \in V(W_i)$. If $v = w_i$, then $\tilde{g}(v) = g(w_i) = \tilde{g}(w)$. Otherwise, since $vw_i \in E(K) \subseteq E(G_1)$, it follows that $\tilde{g}(v) = \tilde{g}(w_i)$ or $\tilde{g}(v)\tilde{g}(w_i) \in E(C_n)$. But since $\tilde{g}(w) = \tilde{g}(w_i)$, this becomes $\tilde{g}(v) = \tilde{g}(w)$ or $\tilde{g}(v)\tilde{g}(w) \in E(C_n)$. This concludes the proof that $g : G_1 \to C_n$ extends to a cyclic map $\tilde{g} : G \to C_n$, and, similarly, any cyclic map $g : G_2 \to C_n$ has such an extension as well. ∎

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